

# Distributed Stochastic Approximation Algorithm With Expanding Truncations: Algorithm and Applications

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**Abstract**—In the paper, a novel distributed stochastic approximation algorithm (DSAA) is proposed to seek roots of the sum of local functions, each of which is associated with an agent from the multiple agents connected in a network. At any time, each agent updates its estimate for the root utilizing the observation of its local function and the information derived from the neighboring agents. The key difference of the proposed algorithm from the existing ones consists in the expanding truncations (so it is called as DSAAWET), by which the boundedness of the estimates can be guaranteed without imposing the growth rate constraints on the local functions. The convergence of the estimates generated by DSAAWET to a consensus set belonging to the root set of the sum function is shown under weaker conditions on the local functions and on the observation noise in comparison with the existing results. We illustrate our approach by two applications, one from signal processing and the other one from distributed optimization. Numerical simulation results are also included.

**Index Terms**—Distributed stochastic approximation, expanding truncation, multi-agent network, distributed optimization.

## I. INTRODUCTION

Distributed algorithms have been extensively investigated in connection with the problems arising from sensor networks and networked systems for recent years, for example, consensus problem [1]–[3], distributed estimation [4], [5], sensor localization [6], distributed optimization [7]–[9], distributed control [10], [11] and so on. The distributed algorithms work in the situation, where the goal is cooperatively accomplished by a multi-agent network with computation and communication abilities allocated in a distributed environment. Their advantages over the centralized approaches for networked problems consist in enhancing the robustness of the networks, preserving privacy, and reducing the communication and computation costs.

Stochastic approximation (SA) was first considered in the work [20] of Robbins and Monro for finding roots of a function with noisy observations, now it is known as the RM algorithm. Then SA was used by Kiefer and Wolfowitz [21] to estimate the maximum of an expectation function only with noisy function observations. SA has found wide applications in signal processing, communications and adaptive control, see,

e.g., [22], [27], [28]. Recently, many distributed problems are solved by SA-based distributed algorithms, e.g., distributed parameter estimation [4], distributed convex optimization over random networks [9], and searching local minima of a non-convex objective function [15]. As a result, investigating the distributed algorithm for SA is also of great importance.

Distributed stochastic approximation algorithms (DSAA) were proposed in [12], [13] to cooperatively find roots of a function, being a sum of local functions associated with agents in a multi-agent network. Each agent updates its estimate for the root based on its local information composed of: 1) the observation of its local function possibly corrupted by noise and 2) the information obtained from its neighbors. The weak convergence for DSAA with constant step-size is investigated in [12], while the almost sure convergence for DSAA with decreasing step-size is studied in [13]. Performance gap between the distributed and the centralized stochastic approximation algorithms is investigated in [14]. However, DSAA discussed in [32]–[34] is in a different setting in comparison with [12], [13]. In fact, all components of the root-vector are estimated by each agent in [12], [13], while in [32]–[34] the components are separately estimated at different processors.

It is noticed that almost all aforementioned SA-based distributed algorithms require rather restrictive conditions to guarantee convergence. For example, in [13] it is required that each local function is globally Lipschitz continuous and the observation noise is a martingale difference sequence (mds). However, these conditions may not hold for some problems, e.g., distributed principle component analysis and distributed gradient-free optimization to be discussed in Section II.B. This paper aims at solving the distributed root-seeking problem under weaker conditions in comparison with those used in [13].

Contributions of the paper are as follows. 1) We propose a novel DSAA with expanding truncations (DSAAWET) for networks with deterministic switching topologies by designing a network truncation mechanism. It is shown that estimates generated by DSAAWET at all agents approach with probability one to a consensus set, which is contained in the root set of the sum function. Compared with [13], we neither assume the observation noise to be an mds nor impose any growth rate constraint and Lipschitz continuity on the local functions. 2) We apply the proposed algorithm to two application problems stated in Section II.B, and establish the corresponding theoretic results. In contrast, the algorithm proposed in [13] is not applicable to these problems.

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TABLE I: Notations

Symbol	Definition
$\ v\ , \ A\ $	Euclidean ( $l^2$ ) norm of vector $v$ , matrix $A$
$A \geq 0$	Each entry of matrix $A$ is nonnegative, and $A$ is called the nonnegative matrix.
$\mathbf{I}_m$	$m \times m$ identity matrix
$\mathbf{1}, \mathbf{0}$	Vector or matrix with all entries equal to 1, 0
$X^T, X^{-1}$	Transpose of matrix $X$ , inverse of matrix $X$
$\text{col}\{x_1, \dots, x_m\}$	$\text{col}\{x_1, \dots, x_m\} \triangleq (x_1^T, \dots, x_m^T)^T$ stacking the vectors or matrices $x_1, \dots, x_m$
$I_{[\text{inequality}]}$	Indicator function meaning that it equals 1 if the inequality indicated in the bracket is fulfilled, and 0 if the inequality does not hold
$\otimes$	Kronecker product
$a \wedge b$	$\min\{a, b\}$
$d(x, \Omega)$	$\inf_y \{\ x - y\  : y \in \Omega\}$
$E[\cdot]$	Expectation operator
$D_\perp$	$D_\perp \triangleq (\mathbf{I}_N - \frac{\mathbf{1}\mathbf{1}^T}{N}) \otimes \mathbf{I}_l$ , $N$ = the number of agents in the network, $l$ = the number of arguments of the local function
$m(k, T)$	$m(k, T) \triangleq \max\{m : \sum_{i=k}^m \gamma_i \leq T\}$ is an integer valued function for $T > 0$ and integer $k$
$\tau_{i,m}$	The smallest time when the truncation number of agent $i$ has reached $m$ , i.e., $\tau_{i,m} \triangleq \inf\{k : \sigma_{i,k} = m\}$ , where $\{\sigma_{i,k}\}$ is generated by (10)-(13).
$\tau_m$	$\tau_m \triangleq \min_{i \in \mathcal{V}} \tau_{i,m}$ , the smallest time when at least one of agents has its truncation number reached $m$
$\tilde{\tau}_{j,m}$	$\tilde{\tau}_{j,m} \triangleq \tau_{j,m} \wedge \tau_{m+1}$
$\hat{\sigma}_{i,k}$	$\hat{\sigma}_{i,k} \triangleq \max_{j \in N_i(k)} \sigma_{j,k}$ , where $N_i(k)$ is the set of neighboring agents of agent $i$ at time $k$ .
$\sigma_k$	The largest truncation number among all agents at time $k$ , i.e., $\sigma_k = \max_{i \in \mathcal{V}} \sigma_{i,k} = \max_{i \in \mathcal{V}} \hat{\sigma}_{i,k}$ .
$\delta(t)$	$\delta(t) \rightarrow 0$ as $t \rightarrow 0$ .

The rest of the paper is arranged as follows. The distributed root-seeking problem is formulated, and two motivation examples are given in Section II. DSAAWET is defined and the corresponding convergent results are presented in Section III. The proof of the main results is given in Section IV with some details placed in Appendices. The proposed algorithm is applied to solve the two application problems in Section V with numerical examples included. Some concluding remarks are given in Section VI.

## II. PROBLEM FORMULATION AND MOTIVATIONS

We first formulate the distributed root-seeking problem with the related communication model. Then we give two motivation problems that cannot be solved by the existing algorithms, but can be solved by DSAAWET to be proposed in the paper.

### A. Problem Statement

Consider the case where all agents in a network collectively search the root of the sum function given by

$$f(\cdot) = \frac{1}{N} \sum_{i=1}^N f_i(\cdot), \quad (1)$$

where  $f_i(\cdot) : \mathbb{R}^l \rightarrow \mathbb{R}^l$  is the local function assigned to agent  $i$  and can only be observed by agent  $i$ . Let  $J \triangleq \{x \in \mathbb{R}^l : f(x) = 0\}$  denote the root set of  $f(\cdot)$ .

For any  $i \in \mathcal{V}$ , denote by  $x_{i,k} \in \mathbb{R}^l$  the estimate for the root of  $f(\cdot)$  given by agent  $i$  at time  $k$ . Agent  $i$  at time  $k+1$  has its local noisy observation

$$O_{i,k+1} = f_i(x_{i,k}) + \varepsilon_{i,k+1}, \quad (2)$$

where  $\varepsilon_{i,k+1}$  is the observation noise. Agent  $i$  is required to update its estimate  $x_{i,k}$  on the basis of its local observation and the information obtained from its neighbors.

The information exchange among the  $N$  agents at time  $k$  is described by a digraph  $\mathcal{G}(k) = \{\mathcal{V}, \mathcal{E}(k)\}$ , where  $\mathcal{V} = \{1, \dots, N\}$  is the node set with node  $i$  representing agent  $i$ ;  $\mathcal{E}(k) \subset \mathcal{V} \times \mathcal{V}$  is the edge set with  $(j, i) \in \mathcal{E}(k)$  if agent  $i$  can get information from agent  $j$  at time  $k$  by assuming  $(i, i) \in \mathcal{E}(k)$ . Let the associated adjacency matrix be denoted by  $W(k) = [\omega_{ij}(k)]_{i,j=1}^N$ , where  $\omega_{ij}(k) > 0$  if and only if  $(j, i) \in \mathcal{E}(k)$ , and  $\omega_{ij}(k) = 0$ , otherwise. Denote by  $N_i(k) = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}(k)\}$  the neighbors (neighboring agents) of agent  $i$  at time  $k$ .

A time-independent digraph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$  is called strongly connected if for any  $i, j \in \mathcal{V}$  there exists a directed path from  $i$  to  $j$ . By this we mean a sequence of edges  $(i, i_1), (i_1, i_2), \dots, (i_{p-1}, j)$  in the digraph with distinct nodes  $i_m \in \mathcal{V} \ \forall m : 0 \leq m \leq p-1$ , where  $p$  is called the length of the directed path. A nonnegative square matrix  $A$  is called doubly stochastic if  $A\mathbf{1} = \mathbf{1}$  and  $\mathbf{1}^T A = \mathbf{1}^T$ , where  $\mathbf{1}$  is the vector of compatible dimension with all entries equal to 1.

### B. Motivation Examples

We now give two motivation examples, and show that they cannot be solved by the existing distributed stochastic approximation algorithm [13]. We will return to these examples in Section V to show that DSAAWET proposed in the paper can solve them.

1) *Distributed Principle Component Analysis*: In signal processing and pattern recognition, effectively cluster large data sets is a major objective. Principle component analysis (PCA) is a powerful technique to process multivariate data by constructing a concise data representation through computing the dominant eigenvalues and the corresponding eigenvectors of the data covariance matrix [36], [37]. It is delivered in [36] that for recent applications, the large data sets are gathered by a group of spatially distributed sensors. So, the centralized

PCA may no longer be applicable since these distributed data sets are often too large to send to a fusion center. Some techniques are developed in [36] to compute the global PCA for distributed data sets with updating, but there is no convergence analysis. We aim to propose a distributed algorithm for the global PCA, and to establish its convergence.

The global data matrix  $u_k$  at time  $k$  is distributed among  $N$  sensors that spatially distributed in the network:

$$u_k = \begin{pmatrix} u_{1,k} \\ \vdots \\ u_{N,k} \end{pmatrix},$$

where  $u_{i,k} \in \mathbb{R}^{p_i \times d}$  is collected by sensor  $i$ . The rows of  $u_k$  denote the observations and columns denote the features. Assume that for each  $i \in \mathcal{V}$ ,  $\{u_{i,k}\}$  is an iid sequence with zero mean. Denote by  $A = E[u_k^T u_k]$  the covariance matrix of  $u_k$ . The primary objective is to estimate the largest eigenvalue and the corresponding eigenvector of  $A$ .

It has been shown in [35] that finding the unit eigenvector corresponding to the largest eigenvalue of  $A$  can be reduced to find the nonzero root of function  $f(x) = Ax - (x^T Ax)x$ . Then the distributed implementation of PCA is converted to finding the nonzero root of the function given by

$$f(x) = \frac{1}{N} \sum_{i=1}^N f_i(x),$$

where  $f_i(x) = A_i x - (x^T A_i x)x$  with  $A_i = E[u_{i,k}^T u_{i,k}]$ .

Denote by  $x_{i,k}$  the estimate given by agent  $i$  at time  $k$  for the unit eigenvector corresponding to the largest eigenvalue of matrix  $A$ . Since  $A_i$  cannot be directly derived, by replacing  $A_i$  with its sample  $A_{i,k} = u_{i,k}^T u_{i,k}$  at time  $k$ , the local observation of agent  $i$  at time  $k+1$  is as follows

$$O_{i,k+1} = A_{i,k} x_{i,k} - (x_{i,k}^T A_{i,k} x_{i,k}) x_{i,k}. \quad (3)$$

Then, the distributed principle component analysis is in the distributed root-seeking form. Noting that  $f_i(\cdot)$  contains a cubic term, the local function is not globally Lipschitz continuous. Consequently, DSAA proposed in [13] cannot be directly applied to this example.

2) *Distributed Gradient-free Optimization:* Consider a multi-agent network of  $N$  agents, for which the objective is to cooperatively solve the following optimization problem

$$\min_x c(x) = \sum_{i=1}^N c_i(x), \quad (4)$$

where  $c_i(\cdot) : \mathbb{R}^l \rightarrow \mathbb{R}$  is the local objective function of agent  $i$ , and  $c_i(\cdot)$  is only known by  $i$  itself. Assume the optimization problem (4) has solutions. Consider the case where gradients of the cost functions are unavailable but the cost functions can be observed with noises. Then the finite time difference of the cost functions can be adopted to estimate the gradient, see [21] [29] [31]. This problem is referred to as the gradient-free optimization [16]. For the distributed optimization problem with a convex set constraint and with each cost function being convex, [17] [18] have developed the distributed zeroth-order methods by only using functional evaluations, and proved that the expected function value sequence converges to the

optimal value. In contrast, we aim at deriving the almost sure convergence without convex set constraint.

Assume that the cost functions  $c_i(\cdot)$ ,  $i = 1, \dots, N$  are differentiable and the global function  $c(\cdot)$  is convex. Then solving the problem (4) is equivalent to finding roots of the function  $f(x) = \sum_{i=1}^N f_i(x)/N$  with  $f_i(\cdot) = -\nabla c_i(\cdot)$ , where  $\nabla c_i(\cdot)$  is the gradient of  $c_i(\cdot)$ . Here the randomized KW method proposed in [29] is adopted to estimate gradients of the cost functions. Denote by  $x_{i,k}$  the estimate given by agent  $i$  for the solution to problem (4) at time  $k$ . Let  $\Delta_{i,k} \in \mathbb{R}^l$ ,  $k = 1, 2, \dots$  be a sequence of mutually independent random vectors with each component independently taking values  $\pm 1$  with probability  $\frac{1}{2}$ . The observation of function  $-\nabla c_i(\cdot)$  at point  $x_{i,k}$  is constructed as follows

$$O_{i,k+1} = -\frac{c_i(x_{i,k}^+) + \xi_{i,k+1}^+ - c_i(x_{i,k}^-) - \xi_{i,k+1}^-}{2\alpha_k} \Delta_{i,k}, \quad (5)$$

where  $x_{i,k}^+ = x_{i,k} + \alpha_k \Delta_{i,k}$ ,  $x_{i,k}^- = x_{i,k} - \alpha_k \Delta_{i,k}$ ,  $\xi_{i,k+1}^+$  and  $\xi_{i,k+1}^-$  are the observation noises of the cost function  $c_i(\cdot)$  at  $x_{i,k}^+$  and  $x_{i,k}^-$ , respectively, and  $\alpha_k > 0$  for any  $k \geq 0$ .

Thus, we have transformed the distributed gradient-free optimization to the distributed root-seeking form. It is shown in [29] that for any  $i \in \mathcal{V}$ , the observation noise  $\varepsilon_{i,k+1} = O_{i,k+1} + \nabla c_i(x_{i,k})$  is not an mds, and hence DSAA proposed in [13] cannot be directly applied to this problem.

The problems arising from these two examples motivate us to propose DSAAWET, by which the two motivation examples are well dealt with as to be shown in Section V.

### III. DSAAWET AND ITS CONVERGENCE

In this section, we define DSAAWET and formulate the main results of the paper.

#### A. DSAAWET

Let us first explain the idea of expanding truncations. In many important problems, the sequence of estimates generated by the RM algorithm may not be bounded, and it is hard to define in advance a region the sought-for parameter belongs to. This motivates us to adaptively define truncation bounds as follows. When the estimate crosses the current truncation bound, the estimate is reinitialized to some pre-specified point  $x^*$  and at the same time the truncation bound is enlarged. Due to the decreasing step-sizes, this operation makes as if we rerun the algorithm with initial value  $x^*$ , with smaller step-sizes and larger truncation bounds. The expanding truncation mechanism incorporating with some verifiable conditions makes the reinitialization cease in a finite number of steps, and hence makes the estimates bounded. As results, after a finite number of steps the algorithm runs as the RM algorithm. This is well explained in [27], [28] and in the related references therein.

We now apply the idea of expanding truncation to distributed estimation, i.e., for the case  $N > 1$ . This leads to DSAAWET (10)-(13) at the bottom of p.4 with initial values  $x_{i,0}$ ,  $i = 1, \dots, N$ , where  $O_{i,k+1}$  defined by (2) is the local observation of agent  $i$ ,  $\{\gamma_k\}_{k \geq 0}$  with  $\gamma_k > 0$  are step-sizes used by all agents,  $x^* \in \mathbb{R}^l$  is a fixed vector known

to all agents,  $\{M_k\}_{k \geq 0}$  is a sequence of positive numbers increasingly diverging to infinity with  $M_0 \geq \|x^*\|$ ,  $\sigma_{i,k}$  is the number of truncations for agent  $i$  up-to-time  $k$ , and  $M_{\hat{\sigma}_{i,k}}$  serves as the truncation bound when the  $(k+1)$ th estimate for agent  $i$  is generated.

The algorithm (10)-(13) is performed according to the following three steps.

i) **Consensus for truncation numbers.** At time  $k$ , the algorithm for agent  $i$  may or may not be truncated. This yields that at time  $k$ ,  $\sigma_{i,k}$  may be different for  $i = 1, \dots, N$ . The truncation number for each agent is expected to achieve consensus, which can be set to the largest truncation number among the agents. Hence it is required to carry out a max consensus procedure [19] on the truncation numbers. Thus, we set the truncation number  $\hat{\sigma}_{i,k}$  to be the largest one among its neighbors  $\{\sigma_{j,k}, j \in \mathcal{N}_i(k)\}$ , as indicated by (10).

ii) **Average consensus + innovation update.** At time  $k+1$ , agent  $i$  produces an intermediate value  $x'_{i,k+1}$  as shown by (11). If  $\sigma_{i,k} < \hat{\sigma}_{i,k}$ , then  $x'_{i,k+1} = x^*$ . Otherwise,  $x'_{i,k+1}$  is a combination of the consensus part and the innovation part, where the consensus part is a weighted average of estimates derived at its neighbors, and the innovation part processes its local current observation.

iii) **Local truncation judgement.** If  $x'_{i,k+1}$  remains inside its local truncation bound  $M_{\hat{\sigma}_{i,k}}$ , i.e.,  $\|x'_{i,k+1}\| \leq M_{\hat{\sigma}_{i,k}}$ , then  $x_{i,k+1} = x'_{i,k+1}$  and  $\sigma_{i,k+1} = \hat{\sigma}_{i,k}$ . If  $x'_{i,k+1}$  exits from the sphere with radius  $M_{\hat{\sigma}_{i,k}}$ , i.e.,  $\|x'_{i,k+1}\| > M_{\hat{\sigma}_{i,k}}$ , then  $x_{i,k+1}$  is pulled back to the pre-specified point  $x^*$ , and at the same time the truncation number  $\sigma_{i,k+1}$  is increased to  $\hat{\sigma}_{i,k} + 1$ . This is described by (12), (13).

Denote by

$$\sigma_k \triangleq \max_{i \in \mathcal{V}} \sigma_{i,k}, \quad (6)$$

the largest truncation number among all agents at time  $k$ . If  $N = 1$ , then by denoting

$$\hat{\sigma}_{i,k} = \sigma_{i,k} \triangleq \sigma_k, \quad x_{i,k} \triangleq x_k, \quad x'_{i,k} \triangleq x'_k, \quad O_{i,k} \triangleq O_k$$

it can be easily seen that (10)-(13) becomes

$$\begin{aligned} x'_{k+1} &= x_k + \gamma_k O_{k+1}, \\ x_{k+1} &= x^* I_{[\|x'_{k+1}\| > M_{\sigma_k}]} + x'_{k+1} I_{[\|x'_{k+1}\| \leq M_{\sigma_k}]}, \\ \sigma_{k+1} &= \sigma_k + I_{[\|x'_{k+1}\| > M_{\sigma_k}]}. \end{aligned}$$

The equation is called the stochastic approximation algorithm with expanding truncations (SAAWET), which requires possibly the weakest conditions for its convergence among various modifications of the RM algorithm [22]–[24], [27]. Thus, the

advantages of SAAWET over the RM algorithm might remain for the case  $N > 1$ .

*Remark 3.1:* It is noticed that  $\sigma_{i,k+1} \geq \hat{\sigma}_{i,k} \geq \sigma_{i,k} \quad \forall k \geq 0$  by (10) and (13). Further, it is concluded that

$$x_{i,k+1} = x^* \text{ if } \sigma_{i,k+1} > \sigma_{i,k}. \quad (14)$$

This can be seen from the following consideration: i) If  $\sigma_{j,k} \leq \sigma_{i,k} \quad \forall j \in N_i(k)$ , then from (10) we derive  $\hat{\sigma}_{i,k} = \sigma_{i,k}$ . Since  $\sigma_{i,k+1} > \sigma_{i,k}$ , by (13) it follows that  $\|x'_{i,k+1}\| > M_{\hat{\sigma}_{i,k}}$ , and hence from (12) we derive  $x_{i,k+1} = x^*$ . ii) If there exists  $j \in N_i(k)$  such that  $\sigma_{j,k} > \sigma_{i,k}$ , then from (10) we derive  $\hat{\sigma}_{i,k} = \max_{j \in N_i(k)} \sigma_{j,k} > \sigma_{i,k}$ , and from (11) we have  $x'_{i,k+1} = x^*$ . Consequently, by (12) we have  $x_{i,k+1} = x^*$ .

## B. Assumptions

We list the assumptions to be used.

A1  $\gamma_k > 0, \gamma_k \xrightarrow[k \rightarrow \infty]{} 0$ , and  $\sum_{k=1}^{\infty} \gamma_k = \infty$ .

A2 There exists a continuously differentiable function  $v(\cdot) : \mathbb{R}^l \rightarrow \mathbb{R}$  such that

$$\sup_{\delta \leq d(x,J) \leq \Delta} f^T(x) v_x(x) < 0 \quad (15)$$

for any  $\Delta > \delta > 0$ , where  $v_x(\cdot)$  denotes the gradient of  $v(\cdot)$  and  $d(x, J) = \inf_y \{\|x - y\| : y \in J\}$ ,

b)  $v(J) \triangleq \{v(x) : x \in J\}$  is nowhere dense,

c)  $\|x^*\| < c_0$  and  $v(x^*) < \inf_{\|x\|=c_0} v(x)$  for some positive constant  $c_0$ , where  $x^*$  is used in (11) (12).

A3 The local functions  $f_i(\cdot) \quad \forall i \in \mathcal{V}$  are continuous.

A4 a)  $W(k) \quad \forall k \geq 0$  are doubly stochastic matrices;

b) There exists a constant  $0 < \eta < 1$  such that

$$\omega_{ij}(k) \geq \eta \quad \forall j \in \mathcal{N}_i(k) \quad \forall i \in \mathcal{V} \quad \forall k \geq 0;$$

c) The digraph  $\mathcal{G}_{\infty} = \{\mathcal{V}, \mathcal{E}_{\infty}\}$  is strongly connected, where

$$\mathcal{E}_{\infty} = \{(j, i) : (j, i) \in \mathcal{E}(k) \text{ for infinitely many indices } k\};$$

d) There exists a positive integer  $B$  such that

$$(j, i) \in \mathcal{E}(k) \cup \mathcal{E}(k+1) \cup \dots \cup \mathcal{E}(k+B-1)$$

for all  $(j, i) \in \mathcal{E}_{\infty}$  and any  $k \geq 0$ .

A5 For any  $i \in \mathcal{V}$ , the noise sequence  $\{\varepsilon_{i,k+1}\}_{k \geq 0}$  satisfies

a)  $\gamma_k \varepsilon_{i,k+1} \xrightarrow[k \rightarrow \infty]{} 0$ , and

$$\lim_{T \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{1}{T} \left\| \sum_{m=n_k}^{m(n_k, t_k)} \gamma_m \varepsilon_{i,m+1} I_{[\|x_{i,m}\| \leq K]} \right\| = 0$$

$$\sigma_{i,0} = 0, \quad \hat{\sigma}_{i,k} \triangleq \max_{j \in N_i(k)} \sigma_{j,k}, \quad (10)$$

$$x'_{i,k+1} = \left( \sum_{j \in N_i(k)} \omega_{ij}(k) (x_{j,k} I_{[\sigma_{j,k} = \hat{\sigma}_{i,k}]} + x^* I_{[\sigma_{j,k} < \hat{\sigma}_{i,k}]} + \gamma_k O_{i,k+1}) \right) I_{[\sigma_{i,k} = \hat{\sigma}_{i,k}]} + x^* I_{[\sigma_{i,k} < \hat{\sigma}_{i,k}]}, \quad (11)$$

$$x_{i,k+1} = x^* I_{[\|x'_{i,k+1}\| > M_{\hat{\sigma}_{i,k}}]} + x'_{i,k+1} I_{[\|x'_{i,k+1}\| \leq M_{\hat{\sigma}_{i,k}}]}, \quad (12)$$

$$\sigma_{i,k+1} = \hat{\sigma}_{i,k} + I_{[\|x'_{i,k+1}\| > M_{\hat{\sigma}_{i,k}}]}, \quad (13)$$

$\forall t_k \in [0, T]$  for any sufficiently large  $K$

along indices  $\{n_k\}$  whenever  $\{x_{i,n_k}\}$  converges, where  $m(k, T) \triangleq \max\{m : \sum_{i=k}^m \gamma_i \leq T\}$ .

Let us explain the conditions. A1 is a standard assumption for stochastic approximation, see [24], [27]. From A2 it is seen that  $v(\cdot)$  is not required to be positive. A2 a) means that  $v(\cdot)$  serves as a Lyapunov function for the differential equation  $\dot{x} = f(x)$ . It is noticed that A2 b) holds if  $J$  is finite, and A2 c) takes place if  $v(\cdot)$  is radially unbounded.

Condition A4 describes the connectivity properties of the communication graphs. For detailed explanations we refer to [7]. Set  $\Phi(k, k+1) = \mathbf{I}_N$ ,  $\Phi(k, s) = W(k) \cdots W(s) \forall k \geq s$ . By [7, Proposition 1] there exist constants  $c > 0$  and  $0 < \rho < 1$  such that

$$\|\Phi(k, s) - \frac{1}{N} \mathbf{1}\mathbf{1}^T\| \leq c\rho^{k-s+1} \quad \forall k \geq s. \quad (16)$$

*Remark 3.2:* It is noticed that A5 b) is convenient for dealing with state-dependent noise. The indicator function  $I_{[\|x_{i,m}\| \leq K]}$  in the condition will be casted away if the observation noise does not depend on the estimates. However, for the state-dependent noise, before establishing the boundedness of  $\{x_{i,k}\}$ , the condition with an indicator function included is easier to be verified. It is worth noting that in A5 b) we do not assume existence of a convergent subsequence of  $\{x_{i,k}\}$  for any  $i$ , we only require A5 b) hold along indices of any convergent subsequence if exists. Verification of A5 b) along convergent subsequences is much easier than that along the whole sequence. If  $\{\varepsilon_{i,k}\}$  can be decomposed into two parts  $\varepsilon_{i,k} = \varepsilon_{i,k}^{(1)} + \varepsilon_{i,k}^{(2)}$  such that  $\sum_{k=0}^{\infty} \gamma_k \varepsilon_{i,k+1}^{(1)} I_{[\|x_{i,k}\| \leq K]} < \infty$  and  $\varepsilon_{i,k}^{(2)} I_{[\|x_{i,k}\| \leq K]} \xrightarrow[k \rightarrow \infty]{} 0$ , then A5 b) holds. So, A5 holds when the observation noise is an iid sequence or an mds with bounded second moments if  $\sum_{k=1}^{\infty} \gamma_k^2 < \infty$ .

### C. Main Results

Define the vectors  $X_k \triangleq \text{col}\{x_{1,k}, \dots, x_{N,k}\}$ ,  $\varepsilon_k \triangleq \text{col}\{\varepsilon_{1,k}, \dots, \varepsilon_{N,k}\}$ ,  $F(X_k) \triangleq \text{col}\{f_1(x_{1,k}), \dots, f_N(x_{N,k})\}$ . Denote by  $X_{\perp,k} \triangleq D_{\perp} X_k$  the disagreement vector of  $X_k$  with  $D_{\perp} \triangleq (\mathbf{I}_N - \frac{11^T}{N}) \otimes \mathbf{I}_l$ , and by  $\bar{x}_k = \frac{1}{N} \sum_{i=1}^N x_{i,k}$  the average of the estimates derived at all agents at time  $k$ .

*Theorem 3.3:* Let  $\{x_{i,k}\}$  be produced by (10)-(13) with an arbitrary initial value  $x_{i,0}$ . Assume A1-A4 hold. Then for any sample path  $\omega$  where A5 holds for all agents, the following assertions take place:

i)  $\{x_{i,k}\}$  is bounded and there exists a positive integer  $k_0$  possibly depending on  $\omega$  such that

$$x_{i,k+1} = \sum_{j \in N_i(k)} \omega_{ij}(k) x_{j,k} + \gamma_k O_{i,k+1} \quad \forall k \geq k_0, \quad (17)$$

or in the compact form:

$$X_{k+1} = (W(k) \otimes \mathbf{I}_l) X_k + \gamma_k (F(X_k) + \varepsilon_{k+1}) \quad \forall k \geq k_0; \quad (18)$$

$$\text{ii) } X_{\perp,k} \xrightarrow[k \rightarrow \infty]{} \mathbf{0} \quad \text{and} \quad d(x_k, J) \xrightarrow[k \rightarrow \infty]{} 0; \quad (19)$$

iii) there exists a connected subset  $J^* \subset J$  such that

$$d(x_k, J^*) \xrightarrow[k \rightarrow \infty]{} 0. \quad (20)$$

The proof of Theorem 3.3 is presented in Section IV. From the proof it is noticed that for deriving i) the condition A5 a) is not required. Theorem 3.3 establishes that the sequence  $\{X_k\}$  is bounded; the algorithm (10)-(13) finally turns to be a RM-based DSAA without truncations; and the estimates given by all agents converge to a consensus set, which is contained in a connected subset of the root set  $J$ , with probability one when A5 holds for almost all sample paths for all agents. As a consequence, if  $J$  is not dense in any connected set, then  $x_k$  converges to a point in  $J$ . However, it is unclear how does  $\{x_k\}$  behave when  $J$  is dense in some connected set. This problem was investigated for the centralized algorithm in [40].

*Remark 3.4:* Compared with [13], we impose weaker conditions on the local functions and on the observation noise. In fact, we only require the local functions be continuous, while conditions ST1 and ST2 in [13, Theorem 3] do not allow the functions to increase faster than linearly. In addition, we do not require the observation noise to be an mds as in [13]. As shown in [27] [28], A5 is probably the weakest requirement for the noise since it is also necessary for convergence whenever the root  $x^0$  of  $f(\cdot)$  is a singleton and  $f(\cdot)$  is continuous at  $x^0$ . Different from the random communication graphs used in [13], here we use the deterministic switching graphs to describe the communication relationships among agents.

## IV. PROOF OF MAIN RESULTS

Prior to analyzing  $\{x_{i,k}\}$ , let us recall the convergence analysis for SAAWET, i.e., DSAAWET with  $N = 1$ . The key step in the analysis is to establish the boundedness of the estimates, or to show that truncations cease in a finite number of steps. If the number of truncations increases unboundedly, then SAAWET is pulled back to a fixed vector  $x^*$  infinitely many times. This produces convergent subsequences from the estimation sequence. Then the condition A5 b) is applicable and it incorporating with A2 yields a contradiction. This proves the boundedness of the estimates.

Let us try to use this approach to prove the boundedness of  $x_k = \frac{1}{N} \sum_{i=1}^N x_{i,k}$  with  $\{x_{i,k}\}$  generated by (10)-(13). In the case  $\sigma_k \xrightarrow[k \rightarrow \infty]{} \infty$ , we have  $\lim_{k \rightarrow \infty} \sigma_{i,k} = \infty \quad \forall i \in \mathcal{V}$  by Corollary 4.3 given below. Then from Remark 3.1 it is known that the estimate  $x_{i,k+1}$  given by agent  $i$  is pulled back to  $x^*$  when the truncation occurs at time  $k+1$ . This means that  $\{x_{i,k}\} \forall i \in \mathcal{V}$  contains convergent subsequences. However,  $\{x_k\}$  may still not contain any convergent subsequence to make A5 b) applicable. This is because truncations may occur at different times for different  $i \in \mathcal{V}$ . Therefore, the conventional approach used for convergence analysis of SAAWET cannot directly be applied to the algorithm (10)-(13).

To overcome the difficulty, we first introduce auxiliary sequences  $\{\tilde{x}_{i,k}\}$  and  $\{\tilde{\varepsilon}_{i,k+1}\}$  for any  $i \in \mathcal{V}$ . It will be shown in Lemma 4.1 that  $\{\tilde{x}_{i,k}\}$  satisfies the recursions (27)-(29), for which the truncation bound at time  $k$  is the same  $M_{\sigma_k}$  for all agents and the estimates  $\tilde{x}_{i,k+1} \forall i \in \mathcal{V}$  are pulled back to  $x^*$  when  $\sigma_{k+1} > \sigma_k$ . As a result, the auxiliary sequence  $\{\tilde{X}_k\}$  has

convergent subsequences, where  $\tilde{X}_k \triangleq \text{col}\{\tilde{x}_{1,k}, \dots, \tilde{x}_{N,k}\}$ . Besides, it will be shown in Lemma 4.5 that the noise  $\{\tilde{\varepsilon}_k\}$  satisfies a condition similar to A5 b) along any convergent subsequence of  $\{\tilde{X}_k\}$ , where  $\tilde{\varepsilon}_k \triangleq \text{col}\{\tilde{\varepsilon}_{1,k}, \dots, \tilde{\varepsilon}_{N,k}\}$ . To borrow the analytical method from the centralized stochastic algorithm, we rewrite the algorithm (27)-(29) in the centralized form (41) with observation noise  $\{\zeta_{m+1}\}$ . By the results given in Lemma 4.1 and Lemma 4.5, it is shown in Lemma 4.7 that the noise sequence  $\{\zeta_{m+1}\}$  satisfies (42) along convergent subsequences, which is similar to A5 b) when  $N = 1$ . Then by algorithm (41) and the noise property (42), we show that the number of truncations for all agents converge to the same finite value, and that  $\{\tilde{x}_{i,k}\}_{i \in \mathcal{V}}$  reach a consensus to the root set. Thus,  $\{x_{i,k}\}$  and  $\{\tilde{x}_{i,k}\}$  coincide in a finite number of steps, and their convergence is equivalent. In the rest of this section, we will demonstrate Theorem 3.3 in details based upon the aforementioned ideas.

#### A. Auxiliary Sequences

Denote by  $\tau_{i,m} \triangleq \inf\{k : \sigma_{i,k} = m\}$  the smallest time when the truncation number of agent  $i$  has reached  $m$ , by  $\tau_m \triangleq \min_{i \in \mathcal{V}} \tau_{i,m}$  the smallest time when at least one agent has its truncation number reached  $m$ . Set  $\tilde{\tau}_{j,m} \triangleq \tau_{j,m} \wedge \tau_{m+1}$ , where  $a \wedge b = \min\{a, b\}$ .

For any  $i \in \mathcal{V}$ , define the auxiliary sequences  $\{\tilde{x}_{i,k}\}_{k \geq 0}$  and  $\{\tilde{\varepsilon}_{i,k+1}\}_{k \geq 0}$  as follows:

$$\tilde{x}_{i,k} \triangleq x^*, \quad \tilde{\varepsilon}_{i,k+1} \triangleq -f_i(x^*) \quad \forall k : \tau_m \leq k < \tilde{\tau}_{i,m}, \quad (21)$$

$$\tilde{x}_{i,k} \triangleq x_{i,k}, \quad \tilde{\varepsilon}_{i,k+1} \triangleq \varepsilon_{i,k+1} \quad \forall k : \tilde{\tau}_{i,m} \leq k < \tau_{m+1}, \quad (22)$$

where  $m$  is an integer.

Note that for the considered  $\omega$  there exists a unique integer  $m \geq 0$  corresponding to an integer  $k \geq 0$  such that  $\tau_m \leq k < \tau_{m+1}$ . By definition  $\tilde{\tau}_{i,m} \leq \tau_{m+1} \forall i \in \mathcal{V}$ . So,  $\{\tilde{x}_{i,k}\}_{k \geq 0}$  and  $\{\tilde{\varepsilon}_{i,k+1}\}_{k \geq 0}$  are uniquely determined by the sequences  $\{x_{i,k}\}_{k \geq 0}$  and  $\{\varepsilon_{i,k+1}\}_{k \geq 0}$ . Besides, for any  $k \in [\tau_m, \tau_{m+1})$ , the following assertions hold:

$$\text{i) } \tilde{x}_{i,k} = x^*, \quad \tilde{\varepsilon}_{i,k+1} = -f_i(x^*) \quad \text{if } \sigma_{i,k} < m; \quad (23)$$

$$\text{ii) } \tilde{x}_{i,k} = x_{i,k}, \quad \tilde{\varepsilon}_{i,k+1} = \varepsilon_{i,k+1} \quad \text{if } \sigma_{i,k} = m; \quad (24)$$

$$\text{iii) } \tilde{x}_{j,k} = x^* \quad \text{if } \sigma_{j,k-1} < m; \quad (25)$$

$$\text{iv) } \tilde{x}_{j,k+1} = x^* \quad \forall j \in \mathcal{V} \quad \text{if } \sigma_{k+1} = m + 1. \quad (26)$$

**Proof:** i) From  $\sigma_{i,k} < m$  we see that the truncation number of agent  $i$  is smaller than  $m$  at time  $k$ , then by the definition of  $\tau_{i,m}$  we derive  $\tau_{i,m} > k$ . Thus,  $\tilde{\tau}_{i,m} = \tau_{i,m} \wedge \tau_{m+1} > k$ , and hence from (21) we conclude (23).

ii) From  $\sigma_{i,k} = m$  by definition we have  $\tau_{i,m} \leq k$ , and hence  $\tilde{\tau}_{i,m} = \tau_{m+1} \wedge \tau_{i,m} = \tau_{i,m} \leq k$ . Then by (22) it is clear that (24) holds.

iii) By  $\tau_m \leq k < \tau_{m+1}$  we see  $\sigma_{j,k} \leq m \forall j \in \mathcal{V}$ . We show (25) separately for the cases  $\sigma_{j,k} = m$  and  $\sigma_{j,k} < m$ . 1) Let us first consider the case  $\sigma_{j,k} = m$ . Since  $\sigma_{j,k-1} < m$  and  $\sigma_{j,k} = m$ , by (14) we obtain  $x_{j,k} = x^*$ . Hence from (24) we see  $\tilde{x}_{j,p} = x_{j,p} = x^*$ . 2) We now consider the case  $\sigma_{j,k} < m$ . By (23) we see  $\tilde{x}_{j,k} = x^*$ , which is the assertion of (25).

iv) From  $k \in [\tau_m, \tau_{m+1})$  we see  $\sigma_k = m$ . Thus from  $\sigma_{k+1} = m + 1$  by definition we derive  $\tau_{m+1} = k + 1$ , and hence  $k + 1 \in [\tau_{m+1}, \tau_{m+2})$ . By  $\sigma_k = m$  we see  $\sigma_{j,k} < m + 1 \forall j \in \mathcal{V}$ , then we derive (26) by (25). ■

**Lemma 4.1:** The sequences  $\{\tilde{x}_{i,k}\}, \{\tilde{\varepsilon}_{i,k+1}\}$  defined by (21) (22) satisfy the following recursions

$$\hat{x}_{i,k+1} \triangleq \sum_{j \in N_i(k)} \omega_{ij}(k) \tilde{x}_{j,k} + \gamma_k(f_i(\tilde{x}_{i,k}) + \tilde{\varepsilon}_{i,k+1}), \quad (27)$$

$$\begin{aligned} \tilde{x}_{i,k+1} &= \hat{x}_{i,k+1} I_{[\|\hat{x}_{j,k+1}\| \leq M_{\sigma_k} \quad \forall j \in \mathcal{V}]} \\ &\quad + x^* I_{[\exists j \in \mathcal{V} \quad \|\hat{x}_{j,k+1}\| > M_{\sigma_k}]}, \end{aligned} \quad (28)$$

$$\sigma_{k+1} = \sigma_k + I_{[\exists j \in \mathcal{V} \quad \|\hat{x}_{j,k+1}\| > M_{\sigma_k}]}, \quad \sigma_0 = 0. \quad (29)$$

**Proof:** The proof is given in Appendix A. ■

Before clarifying the property of the noise sequence  $\{\tilde{\varepsilon}_{i,k+1}\}_{k \geq 0}$ , we need the following lemma.

**Lemma 4.2:** Assume A4 holds. Then

$$\text{i) } \sigma_{j,k+Bd_{i,j}} \geq \sigma_{i,k} \quad \forall j \in \mathcal{V} \quad \forall k \geq 0, \quad (30)$$

where  $d_{i,j}$  is the length of the shortest directed path from  $i$  to  $j$  in  $\mathcal{G}_\infty$ , and  $B$  is the positive integer given in A4 d),

$$\text{ii) } \tilde{\tau}_{j,m} \leq \tau_m + BD \quad \forall j \in \mathcal{V} \quad \text{for } m \geq 1, \quad (31)$$

where  $D \triangleq \max_{i,j \in \mathcal{V}} d_{i,j}$ .

**Proof:** i) Since  $\mathcal{G}_\infty$  is strongly connected by A4 c), for any  $j \in \mathcal{V}$  there exists a sequence of nodes  $i_1, i_2, \dots, i_{d_{i,j}-1}$  such that  $(i, i_1) \in \mathcal{E}_\infty, (i_1, i_2) \in \mathcal{E}_\infty, \dots, (i_{d_{i,j}-1}, j) \in \mathcal{E}_\infty$ .

Noticing that  $(i, i_1) \in \mathcal{E}_\infty$ , by A4 d) we have

$$(i, i_1) \in \mathcal{E}(k) \cup \mathcal{E}(k+1) \cup \dots \cup \mathcal{E}(k+B-1).$$

Therefore, there exists a positive integer  $k' \in [k, k+B-1]$  such that  $(i, i_1) \in \mathcal{E}(k')$ . So,  $i \in N_{i_1}(k')$ , and hence by (10) and (13) we derive

$$\sigma_{i_1,k+B} \geq \sigma_{i_1,k'+1} \geq \hat{\sigma}_{i_1,k'} \geq \sigma_{i,k'} \geq \sigma_{i,k}.$$

Similarly, we have  $\sigma_{i_2,k+2B} \geq \sigma_{i_1,k+B} \geq \sigma_{i,k}$ . Continuing this procedure, we finally reach the inequality (30).

ii) Let  $\tau_m = k_1$  for some  $m \geq 1$ . Then there is an  $i$  such that  $\tau_{i,m} = k_1$ . By (30) we have  $\sigma_{j,k_1+Bd_{i,j}} \geq \sigma_{i,k_1} = m \quad \forall j \in \mathcal{V}$ .

For the case where  $\sigma_{j,k_1+Bd_{i,j}} = m \quad \forall j \in \mathcal{V}$ , we have  $\tau_{j,m} \leq k_1 + Bd_{i,j} \quad \forall j \in \mathcal{V}$ . By noticing  $\tau_m = k_1$ , from here by the definition of  $\tilde{\tau}_{j,m}$  we obtain (31):

$$\tilde{\tau}_{j,m} \leq \tau_{j,m} \leq \tau_m + Bd_{i,j} \leq \tau_m + BD \quad \forall j \in \mathcal{V}.$$

For the case where  $\sigma_{j,k_1+Bd_{i,j}} > m$  for some  $j \in \mathcal{V}$ , we have  $\tau_{m+1} \leq k_1 + Bd_{i,j}$  for some  $j \in \mathcal{V}$ , and hence  $\tau_{m+1} \leq \tau_m + BD$ . Again, by noticing  $\tau_m = k_1$  we obtain (31):

$$\tilde{\tau}_{j,m} \leq \tau_{m+1} \leq \tau_m + BD \quad \forall j \in \mathcal{V}. \quad \blacksquare$$

**Corollary 4.3:** If  $\sigma_k \xrightarrow[k \rightarrow \infty]{} \infty$ , then  $\lim_{k \rightarrow \infty} \sigma_{i,k} = \infty \quad \forall i \in \mathcal{V}$ . This is because there exists an  $i_0 \in \mathcal{V}$  such that  $\sigma_{i_0,k} \xrightarrow[k \rightarrow \infty]{} \infty$ . Then from (30) it follows that  $\sigma_{j,k} \xrightarrow[k \rightarrow \infty]{} \infty \quad \forall j \in \mathcal{V}$ .

**Corollary 4.4:** If  $\{\sigma_k\}$  is bounded, then  $\{\tilde{x}_{i,k}\}$  and  $\{x_{i,k}\}, \{\tilde{\varepsilon}_{i,k}\}$  and  $\{\varepsilon_{i,k}\}$  coincide in a finite number of steps.

The result can be derived by (22) and the following assertion

$$\tau_{\sigma+1} = \infty \text{ when } \lim_{k \rightarrow \infty} \sigma_k = \sigma. \quad (32)$$

We now verify (B.2). Since  $\sigma_k$  is defined as the largest truncation number among all agents at time  $k$ , from  $\lim_{k \rightarrow \infty} \sigma_k = \sigma$  we have  $\sigma_{i,k} \leq \sigma \quad \forall k \geq 0 \quad \forall i \in \mathcal{V}$ . From here by the definition of  $\tau_{i,m}$  it follows that  $\tau_{i,\sigma+1} = \inf\{k : \sigma_{i,k} = \sigma + 1\} = \infty \quad \forall i \in \mathcal{V}$ , and hence  $\tau_{\sigma+1} = \infty$ . Thus, (B.2) holds.

*Lemma 4.5:* Assume A5 b) holds at the sample path  $\omega$  under consideration for all agents. Then for this  $\omega$

$$\lim_{T \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{1}{T} \left\| \sum_{s=n_k}^{m(n_k, t_k) \wedge (\tau_{\sigma_{n_k}} + 1 - 1)} \gamma_s \tilde{\varepsilon}_{s+1} I_{\|\tilde{X}_s\| \leq K} \right\| = 0 \quad \forall t_k \in [0, T] \text{ for sufficiently large } K > 0 \quad (33)$$

along indices  $\{n_k\}$  whenever  $\{\tilde{X}_{n_k}\}$  converges at  $\omega$ .

*Proof:* The proof is shown in Appendix B. ■

### B. Local Properties Along Convergent Subsequences

Set  $\Psi(k, s) \triangleq [D_{\perp}(W(k) \otimes \mathbf{I}_l)][D_{\perp}(W(k-1) \otimes \mathbf{I}_l)] \dots [D_{\perp}(W(s) \otimes \mathbf{I}_l)] \quad \forall k \geq s, \quad \Psi(k-1, k) \triangleq \mathbf{I}_{Nl}$ .

Since the matrices  $W(k) \quad \forall k \geq 1$  are doubly stochastic, by using the rule of Kronecker product

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \quad (34)$$

we conclude that for any  $k \geq s-1$

$$\Psi(k, s) = (\Phi(k, s) - \frac{1}{N} \mathbf{1}\mathbf{1}^T) \otimes \mathbf{I}_l, \quad (35)$$

$$\Psi(k, s) D_{\perp} = (\Phi(k, s) - \frac{1}{N} \mathbf{1}\mathbf{1}^T) \otimes \mathbf{I}_l. \quad (36)$$

The following lemma measures the closeness of the auxiliary sequence  $\{\tilde{X}_k\}$  along its convergent subsequence  $\{\tilde{X}_{n_k}\}$ .

*Lemma 4.6:* Assume A1, A3, A4 hold and that A5 b) holds for all agents at the sample path  $\omega$  under consideration. Let  $\{\tilde{X}_{n_k}\}$  be a convergent subsequence of  $\{\tilde{X}_k\} : \tilde{X}_{n_k} \xrightarrow[k \rightarrow \infty]{} \bar{X}$  at the considered  $\omega$ . Then for this  $\omega$  there is a  $T > 0$  such that for sufficiently large  $k$  and any  $T_k \in [0, T]$

$$\tilde{X}_{m+1} = (W(m) \otimes \mathbf{I}_l) \tilde{X}_m + \gamma_m (F(\tilde{X}_m) + \tilde{\varepsilon}_{m+1}) \quad (37)$$

for any  $m = n_k, \dots, m(n_k, T_k)$ , and

$$\|\tilde{X}_{m+1} - \tilde{X}_{n_k}\| \leq c_1 T_k + M'_0, \quad (38)$$

$$\|\tilde{x}_{m+1} - \bar{x}_{n_k}\| \leq c_2 T_k \quad \forall m : n_k \leq m \leq m(n_k, T_k), \quad (39)$$

where  $\bar{x}_k \triangleq \frac{1}{N} (\mathbf{1}^T \otimes \mathbf{I}_l) \tilde{X}_k = \frac{1}{N} \sum_{i=1}^N \tilde{x}_{i,k}$ , and  $c_0, c_1, M'_0$  are positive constants which may depend on  $\omega$ .

The proof is given in Appendix C.

By multiplying both sides of (37) with  $\frac{1}{N} (\mathbf{1}^T \otimes \mathbf{I}_l)$  from left, by  $\mathbf{1}^T W(m) = \mathbf{1}^T$  and (34) it follows that

$$\begin{aligned} \bar{x}_{m+1} &= \bar{x}_m + \gamma_m f(\bar{x}_m) + (\mathbf{1}^T \otimes \mathbf{I}_l) \gamma_m \tilde{\varepsilon}_{m+1} / N \\ &\quad + \gamma_m \sum_{i=1}^N (f_i(\tilde{x}_{i,m}) - f_i(\bar{x}_m)) / N. \end{aligned} \quad (40)$$

Setting  $e_{i,m+1} \triangleq (f_i(\tilde{x}_{i,m}) - f_i(\bar{x}_m)) / N$ ,  $e_{m+1} \triangleq \sum_{i=1}^N e_{i,m+1}$ , and  $\zeta_{m+1} \triangleq (\mathbf{1}^T \otimes \mathbf{I}_l) \tilde{\varepsilon}_{m+1} / N + e_{m+1}$ , we rewrite (40) in the centralized form as follows:

$$\bar{x}_{m+1} = \bar{x}_m + \gamma_m (f(\bar{x}_m) + \zeta_{m+1}). \quad (41)$$

The following lemma gives the property of the noise sequence  $\{\zeta_{k+1}\}$ . For the proof we refer to Appendix D.

*Lemma 4.7:* Assume that all conditions used in Lemma 4.6 are satisfied. Let  $\{\tilde{X}_{n_k}\}$  be a convergent subsequence with limit  $\bar{X}$  at the considered  $\omega$ . Then for this  $\omega$

$$\lim_{T \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{1}{T} \left\| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s \zeta_{s+1} \right\| = 0 \quad \forall T_k \in [0, T]. \quad (42)$$

The following lemma gives the crossing behavior of  $v(\cdot)$  at the trajectory  $\bar{x}_k$  with respect to a non-empty interval that has no intersection with  $v(J)$ .

*Lemma 4.8:* Assume A1-A4 hold and that A5 b) holds for all agents at a sample path  $\omega$ . Then any nonempty interval  $[\delta_1, \delta_2]$  with  $d([\delta_1, \delta_2], v(J)) > 0$  cannot be crossed by  $\{v(\bar{x}_{n_k}), \dots, v(\bar{x}_{m_k})\}$  infinitely many times with  $\{\|\tilde{X}_{n_k}\|\}$  bounded, where by “crossing  $[\delta_1, \delta_2]$  by  $\{v(\bar{x}_{n_k}), \dots, v(\bar{x}_{m_k})\}$ ” it is meant that  $v(\bar{x}_{n_k}) \leq \delta_1, v(\bar{x}_{m_k}) \geq \delta_2$ , and  $\delta_1 < v(\bar{x}_s) < \delta_2 \quad \forall s : n_k < s < m_k$ .

*Proof:* Assume the converse: for some nonempty interval  $[\delta_1, \delta_2]$  with  $d([\delta_1, \delta_2], v(J)) > 0$ , there are infinitely many crossings  $\{v(\bar{x}_{n_k}), \dots, v(\bar{x}_{m_k})\}$  with  $\{\|\tilde{X}_{n_k}\|\}$  bounded.

By the boundedness of  $\{\|\tilde{X}_{n_k}\|\}$ , we can extract a convergent subsequence still denoted by  $\{\tilde{X}_{n_k}\}$  with  $\lim_{k \rightarrow \infty} \tilde{X}_{n_k} = \bar{X}$ . So,  $\lim_{k \rightarrow \infty} \bar{x}_{n_k} = \bar{x}$  with  $\bar{x} \triangleq \frac{\mathbf{1}^T \otimes \mathbf{I}_l}{N} \bar{X}$ . Setting  $T_k = \gamma_{n_k}$  in (39), we derive  $\|\bar{x}_{n_k+1} - \bar{x}_{n_k}\| \leq c_2 \gamma_{n_k} \xrightarrow[k \rightarrow \infty]{} 0$ . By the definition of crossings  $v(\bar{x}_{n_k}) \leq \delta_1 < v(\bar{x}_{n_k+1})$ , we obtain

$$v(\bar{x}_{n_k}) \xrightarrow[k \rightarrow \infty]{} \delta_1 = v(\bar{x}), \quad d(\bar{x}, J) \triangleq \vartheta > 0. \quad (43)$$

Then by (39) it is seen that

$$d(\bar{x}_s, J) > \frac{\vartheta}{2} \quad \forall s : n_k \leq s \leq m(n_k, t) + 1 \quad (44)$$

for sufficiently small  $t > 0$  and large  $k$ . From (41) we obtain

$$\begin{aligned} v(\bar{x}_{m(n_k, t)+1}) &= v(\bar{x}_{n_k} + \sum_{s=n_k}^{m(n_k, t)} \gamma_s (f(\bar{x}_s) + \zeta_{s+1})) \\ &= v(\bar{x}_{n_k}) + v_x(\xi_k)^T \sum_{s=n_k}^{m(n_k, t)} \gamma_s (f(\bar{x}_s) + \zeta_{s+1}), \end{aligned} \quad (45)$$

where  $\xi_k$  is in-between  $\bar{x}_{n_k}$  and  $\bar{x}_{m(n_k, t)+1}$ . We then rewrite (45) as follows:

$$\begin{aligned} v(\bar{x}_{m(n_k, t)+1}) - v(\bar{x}_{n_k}) &= \sum_{s=n_k}^{m(n_k, t)} \gamma_s v_x(\bar{x}_s)^T f(\bar{x}_s) \\ &\quad + \sum_{s=n_k}^{m(n_k, t)} \gamma_s (v_x(\xi_k) - v_x(\bar{x}_s))^T f(\bar{x}_s) \\ &\quad + v_x(\xi_k)^T \sum_{s=n_k}^{m(n_k, t)} \gamma_s \zeta_{s+1}. \end{aligned} \quad (46)$$

By (15) and (44) there exists a constant  $\alpha_1 > 0$  such that

$$v_x(\bar{x}_s)^T f(\bar{x}_s) \leq -\alpha_1 \quad \forall s : n_k \leq s \leq m(n_k, t)$$

for sufficiently small  $t > 0$  and large  $k$ , and hence

$$\sum_{s=n_k}^{m(n_k, t)} \gamma_s v_x(\bar{x}_s)^T f(\bar{x}_s) \leq -\alpha_1 t. \quad (47)$$

Since  $\{\bar{x}_s : n_k \leq s \leq m(n_k, t)\}$  are bounded, by continuity of  $f(\cdot)$  there exists a constant  $c_6 > 0$  such that

$$\sum_{s=n_k}^{m(n_k, t)} \gamma_s \|f(\bar{x}_s)\| \leq c_6 t. \quad (48)$$

Since  $\xi_k$  is in-between  $\bar{x}_{n_k}$  and  $\bar{x}_{m(n_k, t)+1}$ , by continuity of  $v_x(\cdot)$  and (39) we know that

$$v_x(\xi_k) - v_x(\bar{x}_s) = \delta(t) \quad \forall s : n_k \leq s \leq m(n_k, t), \quad (49)$$

where  $\delta(t) \rightarrow 0$  as  $t \rightarrow 0$ . Then by (48) we derive

$$\begin{aligned} & \sum_{s=n_k}^{m(n_k, t)} \gamma_s (v_x(\xi_k) - v_x(\bar{x}_s))^T f(\bar{x}_s) \\ & \leq \delta(t) \sum_{s=n_k}^{m(n_k, t)} \gamma_s \|f(\bar{x}_s)\| \leq \delta(t)t. \end{aligned} \quad (50)$$

Since  $\bar{x}_{n_k} \xrightarrow[k \rightarrow \infty]{} \bar{x}$ , by continuity of  $v_x(\cdot)$  and (39) it follows that for sufficiently small  $t > 0$  and large  $k$

$$v_x(\bar{x}_s) - v_x(\bar{x}) = o(1) + \delta(t) \quad \forall s : n_k \leq s \leq m(n_k, t),$$

where  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ . Then by (49) we derive

$$v_x(\xi_k) - v_x(\bar{x}) = o(1) + \delta(t) \quad \forall s : n_k \leq s \leq m(n_k, t).$$

Consequently, for sufficiently small  $t > 0$  and large  $k$

$$\begin{aligned} & v_x(\xi_k)^T \sum_{s=n_k}^{m(n_k, t)} \gamma_s \zeta_{s+1} \\ & = [(v_x(\xi_k) - v_x(\bar{x})) + v_x(\bar{x})]^T \sum_{s=n_k}^{m(n_k, t)} \gamma_s \zeta_{s+1} \\ & \leq (o(1) + \delta(t) + \|v_x(\bar{x})\|) \left\| \sum_{s=n_k}^{m(n_k, t)} \gamma_s \zeta_{s+1} \right\|. \end{aligned} \quad (51)$$

Substituting (47) (50) (51) into (46), we obtain

$$\begin{aligned} & v(\bar{x}_{m(n_k, t)+1}) - v(\bar{x}_{n_k}) \leq -\alpha_1 t + \delta(t)t \\ & + (o(1) + \delta(t) + \|v_x(\bar{x})\|) \sum_{s=n_k}^{m(n_k, t)} \gamma_s \zeta_{s+1}. \end{aligned}$$

Then by (43) it follows that

$$\begin{aligned} & \limsup_{k \rightarrow \infty} v(\bar{x}_{m(n_k, t)+1}) \leq \delta_1 - \alpha_1 t + \delta(t)t \\ & + (\delta(t) + \|v_x(\bar{x})\|) \limsup_{k \rightarrow \infty} \left\| \sum_{s=n_k}^{m(n_k, t)} \gamma_s \zeta_{s+1} \right\|, \end{aligned}$$

and hence from (42) we have

$$\limsup_{k \rightarrow \infty} v(\bar{x}_{m(n_k, t)+1}) \leq \delta_1 - \frac{\alpha_1}{2} t \quad (52)$$

for sufficiently small  $t$ .

However, by continuity of  $v_x(\cdot)$  and (39) we know that

$$\lim_{t \rightarrow 0} \max_{n_k \leq m \leq m(n_k, t)} \|v(\bar{x}_{m+1}) - v(\bar{x}_{n_k})\| = 0,$$

which implies that  $m(n_k, t) + 1 < m_k$  for sufficiently small  $t$ . Therefore,  $v(\bar{x}_{m(n_k, t)+1}) \in (\delta_1, \delta_2)$ , which contradicts with (52). Consequently, the converse assumption is not true. The proof is completed. ■

### C. Finiteness of Number of Truncations

**Lemma 4.9:** Let  $\{x_{i,k}\}$  be produced by (10)-(13) with an arbitrary initial value  $x_{i,0}$ . Assume A1, A3, A4, and A5 b) hold.

i) If  $\lim_{k \rightarrow \infty} \sigma_k = \infty$ , then there exists an integer sequence  $\{n_k\}$  such that  $\bar{x}_{n_k} = x^*$ , and  $\{\bar{x}_{n_k}\}$  starting from  $x^*$  crosses the sphere with  $\|x\| = c_0$  infinitely many times, where  $\{\bar{x}_{n_k}\}$  is defined in Lemma 4.6 and  $c_0$  is given in A2 c).

ii) If, in addition, A2 also holds, then there exists a positive integer  $\sigma$  possibly depending on  $\omega$  such that

$$\lim_{k \rightarrow \infty} \sigma_k = \sigma. \quad (53)$$

The proof of the lemma is given in Appendix E. Lemma 4.9 says that the largest truncation number among all agents converges, while the following lemma indicates that the truncation numbers at all agents converge to the same limit.

**Lemma 4.10:** Assume all conditions required by Lemma 4.9 are satisfied. Then there exists a positive integer  $\sigma$  such that

$$\lim_{k \rightarrow \infty} \sigma_{i,k} = \sigma \quad \forall i \in \mathcal{V}. \quad (54)$$

*Proof:* Since all conditions required by Lemma 4.9 hold, (53) holds for some positive integer  $\sigma$ . Thus,

$$\sigma_{i,k} \leq \sigma \quad \forall k \geq 0 \quad \forall i \in \mathcal{V}. \quad (55)$$

From (53) by (B.2) we have  $\tau_{\sigma+1} = \infty$ , and hence  $\tilde{\tau}_{i,\sigma} = \tau_{i,\sigma} \leq BD + \tau_\sigma \quad \forall i \in \mathcal{V}$  by (31). This means that the smallest time when the truncation number of agent  $i$  reaches  $\sigma$  is not larger than  $BD + \tau_\sigma$ . So, the truncation number of agent  $i$  after time  $BD + \tau_\sigma$  is not smaller than  $\sigma$ , i.e.,  $\sigma_{i,k} \geq \sigma \quad \forall k \geq BD + \tau_\sigma \quad \forall i \in \mathcal{V}$ , which incorporating with (55) yields

$$\sigma_{i,k} = \sigma \quad \forall k \geq BD + \tau_\sigma \quad \forall i \in \mathcal{V}. \quad (56)$$

Consequently, (54) holds. ■

### D. Proof of Theorem 3.3

**Proof.** i) By (53) and (56) there is a positive integer  $\sigma$  possibly depending on  $\omega$  such that

$$\hat{\sigma}_{i,k} = \sigma_{i,k} = \sigma \quad \forall k \geq k_0 \triangleq BD + \tau_\sigma \quad \forall i \in \mathcal{V}, \quad (57)$$

and hence by (11)

$$x'_{i,k+1} = \sum_{j \in N_i(k)} \omega_{ij}(k) x_{j,k} + \gamma_k O_{i,k+1} \quad \forall k \geq k_0 \quad \forall i \in \mathcal{V}. \quad (58)$$

By (56) (57) we see  $\sigma_{i,k+1} = \hat{\sigma}_{i,k} = \sigma \quad \forall k \geq k_0 \quad \forall i \in \mathcal{V}$ , and hence  $\|x'_{i,k+1}\| \leq M_\sigma$  by (13) and  $x_{i,k+1} = x'_{i,k+1}$  by (12) for any  $k \geq k_0$  and any  $i \in \mathcal{V}$ . So, we conclude that for any  $i \in \mathcal{V}$ ,  $\{x_{i,k}\}$  is bounded and (17) follows from (58).

ii) By multiplying both sides of (18) with  $D_\perp$  from left we derive

$$X_{\perp,k+1} = D_\perp(W(k) \otimes I_l)X_k + \gamma_k D_\perp(F(X_k) + \varepsilon_{k+1}),$$

and inductively

$$X_{\perp,k+1} = \Psi(k, k_0)X_{k_0} + \sum_{m=k_0}^k \gamma_m \Psi(k-1, m)D_\perp(F(X_m) + \varepsilon_{m+1}) \quad \forall k \geq k_0.$$



Then by (35) (36) we derive

$$\begin{aligned} X_{\perp,k+1} &= [(\Phi(k, k_0) - \frac{1}{N} \mathbf{1} \mathbf{1}^T) \otimes \mathbf{I}_l] X_{k_0} \\ &+ \sum_{m=k_0}^k \gamma_m [(\Phi(k-1, m) - \frac{1}{N} \mathbf{1} \mathbf{1}^T) \otimes \mathbf{I}_l] F(X_m) \\ &+ \sum_{m=k_0}^k \gamma_m [(\Phi(k-1, m) - \frac{1}{N} \mathbf{1} \mathbf{1}^T) \otimes \mathbf{I}_l] \varepsilon_{m+1}. \end{aligned}$$

Therefore, from (16) by continuity of  $F(\cdot)$  and the boundedness of  $\{X_s\}$ , we conclude that there exist positive constants  $c_1, c_2, c_3$  possibly depending on  $\omega$  such that

$$\|X_{\perp,k+1}\| \leq c_1 \rho^{k+1-k_0} + c_2 \sum_{m=k_0}^k \gamma_m \rho^{k-m} + c_3 \sum_{m=k_0}^k \gamma_m \rho^{k-m} \|\varepsilon_{m+1}\| \quad \forall k \geq k_0. \quad (59)$$

Noticing that for any given  $\epsilon > 0$  there exists a positive integer  $k_1$  such that  $\gamma_k \leq \epsilon \quad \forall k \geq k_1$ , we then have

$$\begin{aligned} \sum_{m=0}^k \gamma_m \rho^{k-m} &= \sum_{m=0}^{k_1} \gamma_m \rho^{k-m} + \sum_{m=k_1+1}^k \gamma_m \rho^{k-m} \\ &\leq \rho^{k-k_1} \sum_{m=0}^{k_1} \gamma_m + \epsilon \frac{1}{1-\rho} \xrightarrow[k \rightarrow \infty]{\epsilon \rightarrow 0} 0. \end{aligned}$$

Therefore, the second term at the right-hand side of (59) tends to zero as  $k \rightarrow \infty$ . Similarly, the last term at the right-hand side of (59) also tends to zero since  $\lim_{k \rightarrow \infty} \gamma_k \varepsilon_{k+1} = 0$ . Therefore, by  $0 < \rho < 1$  from (59) we conclude that

$$X_{\perp,k} \xrightarrow[k \rightarrow \infty]{} \mathbf{0}.$$

By i) and Corollary 4.4 we see that  $\{\tilde{x}_{i,k} \mid \forall i \in \mathcal{V}\}$  are bounded for this  $\omega$ , and hence  $\{\bar{x}_k\}$  is bounded. The rest of the proof is similar to that given in [27].

We first show the convergence of  $v(\bar{x}_k)$ . Since

$$v_1 \triangleq \liminf_{k \rightarrow \infty} v(\bar{x}_k) \leq \limsup_{k \rightarrow \infty} v(\bar{x}_k) \triangleq v_2,$$

we want to prove  $v_1 = v_2$ . Assume the converse:  $v_1 < v_2$ . Since  $v(J)$  is nowhere dense, there exists a nonnegative interval  $[\delta_1, \delta_2] \in (v_1, v_2)$  such that  $d([\delta_1, \delta_2], v(J)) > 0$ . Then  $v(\bar{x}_k)$  crosses the interval  $[\delta_1, \delta_2]$  infinitely many times. This contradicts Lemma 4.8. Therefore,  $v_1 = v_2$ , which implies the convergence of  $v(\bar{x}_k)$ .

We then prove  $d(\bar{x}_k, J) \xrightarrow[k \rightarrow \infty]{} 0$ . Assume the converse. Then by the boundedness of  $\{\bar{x}_k\}$  there exists a convergent subsequence  $\bar{x}_{n_k} \xrightarrow[k \rightarrow \infty]{} \bar{x}$  with  $d(\bar{x}, J) \triangleq \vartheta > 0$ . From (39) it follows that for sufficiently small  $t > 0$  and large  $k$

$$d(\bar{x}_s, J) > \frac{\vartheta}{2} \quad \forall s : n_k \leq s \leq m(n_k, t),$$

and hence from (15) there exists a constant  $b > 0$  such that

$$v_x(\bar{x}_s)^T f(\bar{x}_s) < -b \quad \forall s : n_k \leq s \leq m(n_k, t).$$

Thus, similar to the proof for obtaining (52) it is seen that for sufficiently small  $t > 0$

$$\limsup_{k \rightarrow \infty} v(\bar{x}_{m(n_k, t)+1}) \leq v(\bar{x}) - \frac{b}{2} t.$$

This contradicts with the convergence of  $v(\bar{x}_k)$ . Therefore,  $d(\bar{x}_k, J) \xrightarrow[k \rightarrow \infty]{} 0$ , and hence  $d(x_k, J) \xrightarrow[k \rightarrow \infty]{} 0$ .

iii) Assume the converse: i.e.,  $J^*$  is disconnected. Then there exist closed sets  $J_1^*$  and  $J_2^*$  such that  $J^* = J_1^* \cup J_2^*$

and  $d(J_1^*, J_2^*) > 0$ . Define  $\rho = \frac{1}{3}d(J_1^*, J_2^*)$ . Noticing  $d(\bar{x}_k, J^*) \xrightarrow[k \rightarrow \infty]{} 0$ , we know there exists  $k_0$  such that

$$\bar{x}_k \in B(J_1^*, \rho) \cup B(J_2^*, \rho) \quad \forall k \geq k_0, \quad (60)$$

where  $B(A, \rho)$  denotes the  $\rho$ -neighborhood of the set  $A$ .

Define

$$\begin{aligned} n_0 &= \inf\{k > k_0, d(\bar{x}_k, J_1^*) < \rho\}, \\ m_p &= \inf\{k > n_p, d(\bar{x}_k, J_2^*) < \rho\}, \\ n_p &= \inf\{k > m_p, d(\bar{x}_k, J_1^*) < \rho\}, \quad p \geq 0. \end{aligned}$$

By (60) we have  $\bar{x}_{n_p} \in B(J_1^*, \rho)$ ,  $\bar{x}_{n_p+1} \in B(J_2^*, \rho)$  for any  $p \geq 0$ . Then by  $d(J_1^*, J_2^*) = 3\rho$  we derive

$$\|\bar{x}_{n_p} - \bar{x}_{n_p+1}\| > \rho. \quad (61)$$

Since  $\{\bar{x}_{n_k}\}$  is bounded, we can extract a convergent subsequence, still denoted by  $\{\bar{x}_{n_k}\}$ . By setting  $T_k = \gamma_{n_k}$  in (39) we derive  $\|\bar{x}_{n_{k+1}} - \bar{x}_{n_k}\| \leq c_2 \gamma_{n_k} \xrightarrow[k \rightarrow \infty]{} 0$ , which contradicts with (61). So, the converse assumption is not true. Hence the proof is completed. ■

## V. CONVERGENCE FOR APPLICATION PROBLEMS

In this section, we establish convergence results for the two problems stated in Section II, and present the corresponding numerical simulations.

### A. Distributed PCA

We now apply DSAAWET to the problem of distributed principle component analysis, and establish its convergence.

**Theorem 5.1:** Let  $\{x_{i,k}\}$  be produced by (10)–(13) with  $O_{i,k+1}$  defined by (3) and  $x_{i,0} = \mathbf{1}/\sqrt{N}$ , where  $x^* = \mathbf{1}/\sqrt{N}$ . Assume that A4 holds, and, in addition, that

$$\text{B1 } \gamma_k > 0, \quad \sum_{k=1}^{\infty} \gamma_k = \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} \gamma_k^2 < \infty;$$

B2 i)  $\{u_{i,k}\}$  is an iid sequence with zero mean and with bounded fourth moment for any  $i \in \mathcal{V}$ ;

ii) the largest eigenvalue of  $A$  is with unit multiplicity with the corresponding eigenvectors denoted by  $u^{(0)}$  and  $-u^{(0)}$ .

Then

$$X_{\perp,k} \xrightarrow[k \rightarrow \infty]{} \mathbf{0} \quad \text{and} \quad d(x_k, J) \xrightarrow[k \rightarrow \infty]{} 0 \quad a.s., \quad (62)$$

where  $J \triangleq \{0, u^{(0)}, -u^{(0)}\}$ . Moreover, for any  $i, j \in \mathcal{V}$

$$\lim_{k \rightarrow \infty} x_{i,k} = \lim_{k \rightarrow \infty} x_{j,k} = u^{(0)} \text{ or } -u^{(0)} \text{ or } \mathbf{0} \quad a.s. \quad (63)$$

**Proof:** We prove this result by adopting the similar procedures as that used in the proof of Theorem 3.3. Firstly, we use the auxiliary sequences defined in (21)(22) to prove the finiteness of truncation numbers. Then we show that the estimates are bounded and finally reach consensus. At last, we show that the estimates either converge to the unit eigenvector corresponding to the largest eigenvalue or to zero.

By B1 the step size  $\{\gamma_k\}$  satisfies A1. Since  $f_i(x) = A_i x - (x^T A_i x)x$ , we see A3 holds. By (2)(3) we know

$$\begin{aligned} \varepsilon_{i,k+1} &= O_{i,k+1} - f_i(x_{i,k}) \\ &= (A_{i,k} - A_i)x_{i,k} - (x_{i,k}^T (A_{i,k} - A_i)x_{i,k})x_{i,k}. \end{aligned}$$

Then from B2 i) we conclude that  $\{\varepsilon_{i,k+1} I[\|x_{i,k}\| \leq K]\}$  is an mds with bounded second moments for any  $K > 0$ , and hence by the convergence theorem for mds [39]

$$\sum_{k=0}^{\infty} \gamma_k \varepsilon_{i,k} I[\|x_{i,k}\| \leq K] < \infty \text{ a.s.} \quad (64)$$

So, A5 b) holds almost surely for any agent  $i$ .

By B2 ii) we see that the nonzero roots of  $f(x) = Ax - (x^T Ax)x$  are  $J^0 = \{u^{(0)}, -u^{(0)}\}$ . Define

$$v(x) = \frac{e^{\|x\|^2}}{x^T Ax} \quad \forall x \neq 0.$$

Then

$$v_x(x) = 2xv(x) - \frac{e^{\|x\|^2} 2Ax}{(x^T Ax)^2} = \frac{2v(x)}{x^T Ax} ((x^T Ax)x - Ax),$$

and hence

$$f^T(x)v_x(x) = -\frac{2v(x)}{x^T Ax} \|f(x)\|^2 < 0 \quad \forall x \notin J. \quad (65)$$

We now prove that  $\lim_{k \rightarrow \infty} \sigma_k = \sigma < \infty$ .

Assume the converse  $\lim_{k \rightarrow \infty} \sigma_k = \infty$ .

By Lemma 4.9 i)  $\{\tilde{X}_k\}$  has a convergent subsequence  $\{\tilde{X}_{n_k}\}$  with  $\tilde{X}_{n_k} = (1 \otimes I_m)x^*$ . Since A1, A3, A4 and A5 b) hold, Lemma 4.6 and 4.7 take place. By noticing  $\bar{x}_{n_k} = 1/\sqrt{N}$ , from (39) we see that for sufficiently small  $T$  and large  $k$ , all  $\bar{x}_{m+1} \quad \forall m : n_k \leq m \leq m(n_k, T)$  are uniformly above zero. This together with (65) yields (47) when we follow the proof procedure for Lemma 4.8 with  $v(J)$  replaced by  $v(J^0)$ . It is worth noting that  $v(J)$  is not defined since  $v(x)$  is not defined at  $x = 0$ , while  $v(J^0)$  is well defined. Then, any nonempty interval  $[\delta_1, \delta_2]$  with  $d([\delta_1, \delta_2], v(J^0)) > 0$  cannot be crossed by  $\{v(\bar{x}_{n_k}), \dots, v(\bar{x}_{m_k})\}$  infinitely many times.

Since  $v(x)$  is radially unbounded, there exists  $c_0 > 1$  such that  $v(x^*) < \inf_{\|x\|=c_0} v(x)$ . By Lemma 4.9 i)  $\{\bar{x}_{n_k}\}$  starting from  $x^*$  crosses the sphere with  $\|x\| = c_0$  infinitely many times. Since  $v(J^0)$  is nowhere dense, there exists a nonempty interval  $[\delta_1, \delta_2] \in (v(x^*), \inf_{\|x\|=c_0} v(x))$  with  $d([\delta_1, \delta_2], v(J^0)) > 0$ . Therefore,  $[\delta_1, \delta_2]$  with  $d([\delta_1, \delta_2], v(J^0)) > 0$  is crossed by  $\{v(\bar{x}_{n_k}), \dots, v(\bar{x}_{m_k})\}$  infinitely many times. This yields a contradiction, hence the converse assumption is not true. Thus,

$$\lim_{k \rightarrow \infty} \sigma_k = \sigma < \infty. \quad (66)$$

Since (66) holds, the proof for Theorem 3.3 i) is still applicable. So, we conclude that  $\{X_k\}$  is bounded, and there exists a positive integer  $k_0$  such that (17) holds. Then by (64)

$$\sum_{k=0}^{\infty} \gamma_k \varepsilon_{i,k} < \infty \text{ a.s.}$$

and A5 holds. Since the proof for the first result in Theorem 3.3 ii) still holds, we derive  $X_{\perp,k} \xrightarrow[k \rightarrow \infty]{} 0$  a.s.

We now prove  $d(x_k, J) \xrightarrow[k \rightarrow \infty]{} 0$  a.s. and (63) by considering the following two cases:

Case 1: If  $X_k$  converges to zero a.s., then by the definition of  $x_k$  it follows that  $x_k \xrightarrow[k \rightarrow \infty]{} 0 \in J$  a.s.

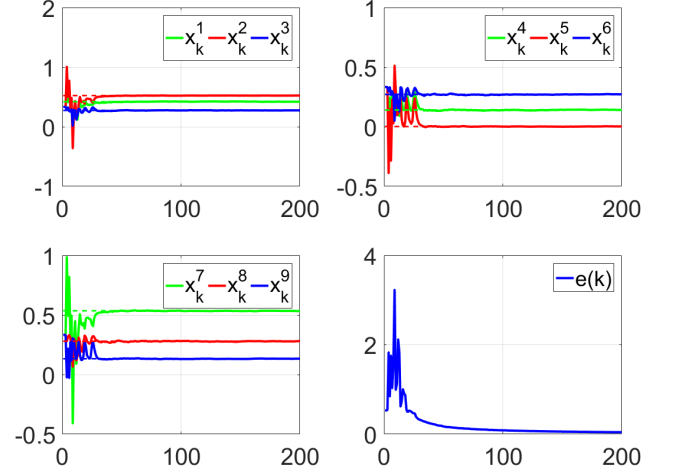


Fig. 1: The estimation sequences and estimation error

Case 2: If  $X_k$  does not converge to zero, then for sufficiently large  $k_1 > k_0$

$$X_k \neq 0 \quad \forall k \geq k_1. \quad (67)$$

This is because the converse assumption that  $X_{k_2} = 0$  for some  $k_2 > k_1$  leads to  $X_k = 0 \quad \forall k \geq k_2$  by (3) and (17). This contradicts with the assumption that  $X_k$  does not converge to zero. So, (67) holds.

By (67) and  $X_{\perp,k} \xrightarrow[k \rightarrow \infty]{} 0$  we see that  $x_k \neq 0$  for sufficiently large  $k$ . Note that  $\{x_{i,k}\}$  and  $\{\tilde{x}_{i,k}\}$  coincide in a finite number of steps by Corollary 4.4 and (66). Then  $\bar{x}_k \neq 0$  for sufficiently large  $k$ . Without loss of generality, we may assume  $\bar{x}_k \neq 0 \quad \forall k \geq 0$ . Since A1, A3, A4, A5 and (65) hold, by the similar proof as that for Theorem 3.3 ii) we derive  $\lim_{k \rightarrow \infty} d(x_k, J^0) = 0$ . Since  $J^0$  is composed of isolated points, we conclude that in this case  $x_k$  converges to either  $u^{(0)}$  or  $-u^{(0)}$ .

Thus, all assertions of the theorem have been proved. ■

**Example 5.2: Distributed Principle Componet Analysis**

Let  $N = 1000$ . The matrix  $W(k)$  is as follows:

$$W(3k-2) = \begin{pmatrix} W_1 & 0 \\ 0 & I_{N/2} \end{pmatrix}, \quad W(3k-1) = \begin{pmatrix} I_{N/2} & 0 \\ 0 & W_2 \end{pmatrix},$$

$$W(3k) = \begin{pmatrix} \frac{1}{2} I_{N/2} & \frac{1}{2} I_{N/2} \\ \frac{1}{2} I_{N/2} & \frac{1}{2} I_{N/2} \end{pmatrix},$$

where matrices  $W_1 \in \mathbb{R}^{N/2 \times N/2}$  and  $W_2 \in \mathbb{R}^{N/2 \times N/2}$  are doubly stochastic. Further, they are the adjacency matrices of some strongly connected digraphs. Thus, A4 holds.

Each sensor  $i = 1, 2, \dots, N$  has access to an 9-dimensional iid Gaussian sequence  $\{u_{i,k}\}$  with zero mean. Set  $\gamma_k = \frac{1}{k}$ ,  $M_k = 2^k$ . Let the sequence  $\{x_{i,k}\}$  be produced by (10)–(13) with  $O_{i,k+1}$  defined by (3) and with  $x_{i,0} = x^* = 1/\sqrt{N}$ . Denote by  $x_k^i$  the  $i$ th component of  $x_k = \frac{1}{N} \sum_{i=1}^N x_{i,k}$ , and by  $e(k) = \sum_{i=1}^N \|x_{i,k} - u^{(0)}\|_2 / N$  the average of 2-norm errors for all agents at time  $k$ .

The estimates  $x_k$  are demonstrated in Fig. 1 with the dashed lines denoting the true values and the real lines the corresponding estimates. The estimation errors  $e(k)$  are demonstrated in

TABLE II: Estimates produced by DSAA in [13]

$k$	0	1	2	3	4
$x_k^1$	1/3	1.9	-587	$5.46 * 10^{11}$	$-1.49 * 10^{41}$
$x_k^2$	1/3	3.2	-593	$3.64 * 10^{11}$	$-9.89 * 10^{40}$
$x_k^3$	1/3	0.277	-45.4	$-9.9 * 10^{10}$	$2.93 * 10^{40}$
$x_k^4$	1/3	-1.26	402	$-6 * 10^{11}$	$1.72 * 10^{41}$
$x_k^5$	1/3	-3.26	736	$-2.23 * 10^{11}$	$4.76 * 10^{40}$
$x_k^6$	1/3	0.25	-17.8	$7.86 * 10^{10}$	$-2.23 * 10^{40}$
$x_k^7$	1/3	3.3	-711	$5.02 * 10^{11}$	$-1.34 * 10^{41}$
$x_k^8$	1/3	0.19	2.48	$-1.76 * 10^{11}$	$5.54 * 10^{40}$
$x_k^9$	1/3	-1.63	397	$-2.37 * 10^{11}$	$6.38^{40}$

Fig. 1. From the figure it is seen that the estimates converge to the unit eigenvector corresponding to the largest eigenvalue of  $A$ .

For a comparison, we also present the simulation results computed by DSAA proposed in [13] for the same example. The initial value for each agent,  $\alpha_k$  and  $\gamma_k$  are set to be the same as those used in DSAWET. The simulation results are given in Table II, from which it is seen that the estimates are unbounded. When a smaller step size, say  $\gamma_k = \frac{1}{k+20}$ , is taken, then the estimates are bounded for some sample paths and are unbounded for others. It is hard to indicate how small the step-sizes should be for a given sample path in advance. As a matter of fact, DSAWET is an adaptive method for choosing step-sizes for any sample path.

### B. Distributed Gradient-free Optimization

We now consider the convergence of DSAWET applied to the distributed gradient-free optimization problem.

**Theorem 5.3:** Let  $\{x_{i,k}\}$  be produced by (10)-(13) with  $O_{i,k+1}$  given by (5) for any initial value  $x_{i,0}$ . Assume that A4 holds, and, in addition,

C1 i)  $\gamma_k > 0$ ,  $\sum_{k=1}^{\infty} \gamma_k = \infty$ , and  $\sum_{k=1}^{\infty} \gamma_k^p < \infty$  for some  $p \in (1, 2]$ ;

ii)  $\alpha_k > 0$ , and  $\alpha_k \rightarrow 0$  as  $k \rightarrow \infty$ ;

iii)  $\sum_{k=1}^{\infty} \gamma_k^2 / \alpha_k^2 < \infty$ ;

C2 i)  $c_i(\cdot)$ ,  $i = 1, 2, \dots, N$  are continuously differentiable, and  $\nabla c(\cdot)$  is locally Lipschitz continuous;

ii)  $c(\cdot)$  is convex with a unique global minimum  $x_{min}$ ;

iii)  $c(x^*) < \sup_{\|x\|=c_0} c(x)$  and  $\|x^*\| < c_0$  for some positive constant  $c_0$ , where  $x^*$  is used in (11) (12);

C3  $\xi_{i,k+1} = \xi_{i,k+1}^+ - \xi_{i,k+1}^-$  is independent of  $\{\Delta_{i,s}, s = 1, 2, \dots, k\}$  for any  $k \geq 0$ , and  $\xi_{i,k+1}$  satisfies one of the following two conditions:

i)  $\sup_k |\xi_{i,k}| \leq \xi_i$  a.s., where  $\xi_i$  may be random;

ii)  $\sup_k E[\xi_{i,k+1}^2] < \infty$

for any  $i = 1, 2, \dots, N$ . Then for any  $i = 1, 2, \dots, N$

$$x_{i,k} \xrightarrow[k \rightarrow \infty]{} x_{min} \text{ a.s.} \quad (68)$$

*Proof:* To apply Theorem 3.3 to this problem, we have to verify conditions A1-A5.

By C1 the step-size  $\{\gamma_k\}$  satisfies A1. Since  $c(\cdot)$  is convex and differentiable,  $x_{min}$  is the global minimum of  $c(\cdot)$  if and only if  $\nabla c(x_{min}) = 0$ . So, the original problem (4) is equivalent to finding the root  $J = \{x_{min}\}$  of  $f(\cdot) = \sum_{i=1}^N f_i(\cdot)/N$  with  $f_i(\cdot) = -\nabla c_i(\cdot)$ . By setting  $v(\cdot) = c(\cdot)$ , we derive  $f^T(x)v_x(x) = -\|\nabla c(x)\|^2/N$  and  $v(J) = \{c(x_{min})\}$ . So,

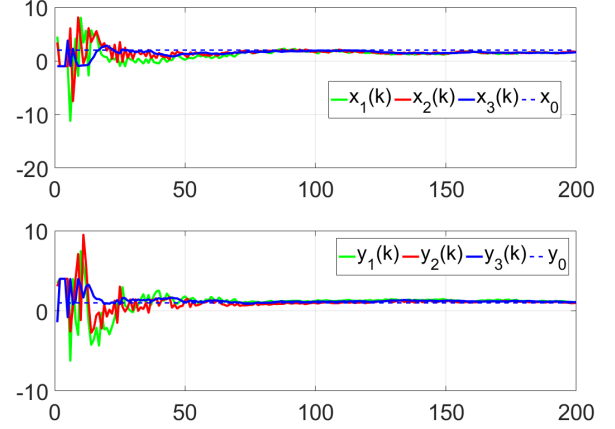


Fig. 2: Trajectories of the estimates

A2 a) and A2 b) hold. By C2 iii) we have A2 c). It is clear that C2 i) implies A3.

By (2) we see

$$\varepsilon_{i,k+1} = O_{i,k+1} + \nabla c_i(x_{i,k}), \quad (69)$$

where  $O_{i,k+1}$  is given by (5). The analysis for the observation noise  $\{\varepsilon_{i,k+1}\}$  is the same as that given in [29], so, is omitted here. It is shown in [29] that  $\{\varepsilon_{i,k+1}\}$  is not an mds for any  $i \in \mathcal{V}$ , but the following limit takes place:

$$\lim_{T \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{1}{T} \left\| \sum_{n=k}^{m(k,t_k)} \gamma_n \varepsilon_{i,n+1} I_{\{\|x_{i,n}\| \leq K\}} \right\| = 0 \quad (70)$$

$\forall t_k \in [0, T]$  for any positive integer  $K$ , a.s.

Therefore, A5 b) holds with probability one for any  $i \in \mathcal{V}$ . By Theorem 3.3 i) we see that  $\{x_{i,k}\}$  is bounded almost surely. So, A5 a) holds by taking  $t_k = \gamma_k$  in (70).

In summary, we have shown that A1-A5 hold. Since  $J = \{x_{min}\}$ , by Theorem 3.3 ii) we derive (68). ■

**Remark 5.4:** If the convex function  $c(\cdot)$  is allowed to have non-unique minima while other conditions in Theorem 5.3 remain unchanged, then by Theorem 3.3 we know that  $x_{i,k} \forall i = 1, \dots, N$  converge to a connected subset  $J^* \subset J$ , where  $J = \{x \in \mathbb{R}^l : \nabla c(x) = 0\}$ . Compared with [17] [18], the almost sure convergence is established here without assuming that each cost function is convex. However, we require the cost functions to be differentiable and only consider the unconstrained optimization problem.

#### Example 5.5: Distributed Gradient-free Optimization

Consider the network of three agents with local cost functions given by

$$L_1(x, y) = x^2 + y^2 + 10 \sin(x);$$

$$L_2(x, y) = (x - 4)^2 + (y - 1)^2 - 10 \sin(x);$$

$$L_3(x, y) = 0.01(x - 2)^4 + (y - 2)^2.$$

Let the communication relationship among the agents be described by a strongly connected digraph with the adjacency matrix being doubly stochastic. The task of the network is to find the minimum  $(x^0, y^0) = (2, 1)$  of the cost function

$L(x, y) = \sum_{i=1}^3 L_i(x, y)$ . Though each local cost function is non-convex, the global cost function  $L(x, y)$  is convex.

Let the observation noise for the cost function of each agent be a sequence of iid random vectors  $\in \mathcal{N}(0, I)$ . The first and second component of the initial values for all agents are set to be mutually independent and uniformly distributed over the intervals  $[-2, 6]$  and  $[-2, 4]$ , respectively. Set  $x^* = (-1, 4)^T$ ,  $\gamma_k = \frac{2}{k}$ ,  $\alpha_k = \frac{1}{k^{0.2}}$ , and  $M_k = 2^k$ . Let the estimates for the minimum be produced by (10)–(13) with  $O_{i,k+1}$  defined by (5). The estimates of  $x^0$  and  $y^0$  produced by the three agents are demonstrated in Fig. 2, where  $x_i(k)$  and  $y_i(k)$  denote agent  $i$ 's estimates for  $x^0$  and  $y^0$  at time  $k$ , respectively. From the figure it is seen that the estimates given by all agents converge to the minimum, which is consistent with the theoretic result.

## VI. CONCLUDING REMARKS

In this paper, DSAAWET is defined to solve the formulated distributed root-seeking problem. The estimates are shown to converge to a consensus set belonging to a connected subset of the root set. Two problems as examples of those which can be solved by DSAAWET are demonstrated with numerical simulations provided.

For further research it is of interest to analyze the convergent rate of the proposed algorithm, and to consider the convergent properties of DSAAWET over random networks taking into account the possible packet loss in communication. It is also of interest to consider the possibility of removing the continuity assumption on  $f_i(\cdot)$ , since for the centralized SAAWET the function is only required be measurable and locally bounded.

## APPENDIX A PROOF LEMMA 4.1

We prove the lemma by induction.

We first prove (27)–(29) for  $k = 0$ . Since  $0 \in [\tau_0, \tau_1)$  and  $\sigma_{i,0} = 0 \forall i \in \mathcal{V}$ , by (24) we derive  $\tilde{x}_{i,0} = x_{i,0}$ ,  $\tilde{\varepsilon}_{i,1} = \varepsilon_{i,1} \forall i \in \mathcal{V}$ . Then by noticing  $\hat{\sigma}_{i,0} = \sigma_{i,0} = 0 \forall i \in \mathcal{V}$ , from (11) (27) we see

$$\tilde{x}_{i,1} = x'_{i,1} \quad \forall i \in \mathcal{V}. \quad (\text{A.1})$$

We now show that  $\tilde{x}_{i,1}$  and  $\sigma_1$  generated by (27)–(29) are consistent with their definitions (21) (22) (6) by considering the following two cases:

i) There is no truncation at  $k = 1$ , i.e.,  $\sigma_{i,1} = 0 \forall i \in \mathcal{V}$ . In this case, from  $\hat{\sigma}_{i,0} = 0 \forall i \in \mathcal{V}$  by (13) we have  $\|x'_{i,1}\| \leq M_0 \forall i \in \mathcal{V}$ , and hence  $\|\tilde{x}_{i,1}\| \leq M_0 \forall i \in \mathcal{V}$  by (A.1). Then  $x_{i,1} = x'_{i,1}$  by (12),  $\tilde{x}_{i,1} = \hat{x}_{i,1}$  and  $\sigma_1 = 0$  by (28) and (29), respectively. These together with (A.1) imply that  $\tilde{x}_{i,1} = x_{i,1} \forall i \in \mathcal{V}$ , which is consistent with the definition of  $\tilde{x}_{i,1}$  given by (22) since  $\tilde{\tau}_{i,0} \leq 1 < \tau_1$ . By (6) we see  $\sigma_1 = \max_{i \in \mathcal{V}} \sigma_{i,1} = 0$ , which is consistent with that derived from (27)–(29).

ii) There is a truncation at  $k = 1$  for some agent  $i_0$ , i.e.,  $\sigma_{i_0,1} = 1$ . In this case, by (12) (13) we derive  $x_{i_0,1} = x^*$ ,  $\|x'_{i_0,1}\| > M_0$ , and hence  $\|\hat{x}_{i_0,1}\| > M_0$  by (A.1). Therefore,  $\tilde{x}_{i,1} = x^* \forall i \in \mathcal{V}$  and  $\sigma_1 = 1$  by (28) (29). By (6) from  $\sigma_{i_0,1} = 1$  we derive  $\sigma_1 = \max_{i \in \mathcal{V}} \sigma_{i,1} = 1$ . Since  $0 \in [\tau_0, \tau_1)$  and  $\sigma_1 = 1$ , by (26) we see  $\tilde{x}_{i,1} = x^* \forall i \in \mathcal{V}$ .

Thus,  $\tilde{x}_{i,1}$  and  $\sigma_1$  defined by (21) (22) and (6) are consistent with those produced by (27)–(29).

In summary, we have proved the lemma for  $k = 0$ .

Inductively, we assume (27)–(29) hold for  $k = 0, 1, \dots, p$ . At the fixed sample path  $\omega$  for a given integer  $p$  there exists a unique integer  $m$  such that  $\tau_m \leq p < \tau_{m+1}$ . We intend to show (27)–(29) also hold for  $k = p + 1$ .

Before doing this, we first express  $\hat{x}_{i,p+1} \forall i \in \mathcal{V}$  produced by (27) for the following two cases:

Case 1:  $\sigma_{i,p} < m$ . Since  $p \in [\tau_m, \tau_{m+1})$ , by (23) we see

$$\tilde{x}_{i,p} = x^*, \quad \tilde{\varepsilon}_{i,p+1} = -f_i(x^*). \quad (\text{A.2})$$

From  $\sigma_{i,p} < m$  it follows that  $\sigma_{j,p-1} < m \forall j \in N_i(p)$ , because otherwise, there would exist a  $j_1 \in N_i(p)$  such that  $\sigma_{j_1,p-1} \geq m$ . Hence by (10) (13) we would derive  $\sigma_{i,p} \geq \hat{\sigma}_{i,p-1} \geq \sigma_{j_1,p-1} \geq m$ , yielding a contradiction.

From  $\sigma_{j,p-1} < m \forall j \in N_i(p)$  by (25) we have  $\tilde{x}_{j,p} = x^* \forall j \in N_i(p)$ , which incorporating with (27) (A.2) yields

$$\hat{x}_{i,p+1} = x^* \quad \forall i : \sigma_{i,p} < m. \quad (\text{A.3})$$

Case 2:  $\sigma_{i,p} = m$ . By  $\tau_m \leq p < \tau_{m+1}$  we see  $\sigma_{j,p} \leq m \forall j \in \mathcal{V}$ , and hence by (10) we obtain

$$\hat{\sigma}_{i,p} = m \quad \forall i : \sigma_{i,p} = m. \quad (\text{A.4})$$

Then by (11)

$$\begin{aligned} x'_{i,p+1} &= \sum_{j \in N_i(p)} \omega_{ij}(p) (x_{j,p} I_{[\sigma_{j,p}=m]} + x^* I_{[\sigma_{j,p}<m]}) \\ &\quad + \gamma_p (f_i(x_{i,p}) + \varepsilon_{i,p+1}). \end{aligned} \quad (\text{A.5})$$

From  $\sigma_{i,p} = m$  and  $p \in [\tau_m, \tau_{m+1})$ , by (24) it is clear that

$$\tilde{x}_{i,p} = x_{i,p}, \quad \tilde{\varepsilon}_{i,p+1} = \varepsilon_{i,p+1}. \quad (\text{A.6})$$

By the first term in (23) and (24), for any  $j \in N_i(p)$  we have

$$\tilde{x}_{j,p} = x_{j,p} \text{ if } \sigma_{j,p} = m, \quad \tilde{x}_{j,p} = x^* \text{ if } \sigma_{j,p} < m. \quad (\text{A.7})$$

Substituting (A.6) (A.7) into (27), from (A.5) we derive

$$\hat{x}_{i,p+1} = x'_{i,p+1} \quad \forall i : \sigma_{i,p} = m. \quad (\text{A.8})$$

Since  $\tau_m \leq p < \tau_{m+1}$ , from  $p < \tau_{m+1}$  it follows that  $\sigma_p < m + 1$ , and hence  $\sigma_{p+1} \leq m + 1$ , while from  $\tau_m \leq p$  it follows that  $\sigma_p = m$ , and hence  $\sigma_{p+1} \geq m$ . Thus, we have  $m \leq \sigma_{p+1} \leq m + 1$ .

We now show that  $\tilde{x}_{i,p+1}$  and  $\sigma_{p+1}$  generated by (27)–(29) are consistent with their definitions (21) (22) (6). We prove this separately for the cases  $\sigma_{p+1} = m + 1$  and  $\sigma_{p+1} = m$ .

Case 1:  $\sigma_{p+1} = m + 1$ . We first show

$$\sigma_{i,p+1} \leq m \text{ if } \sigma_{i,p} < m \quad (\text{A.9})$$

separately for the following two cases 1) and 2): 1)  $\sigma_{i,p} < m$  and  $\sigma_{j,p} < m \forall j \in N_i(p)$ . For this case by (10) we derive  $\hat{\sigma}_{i,p} < m$ , and hence  $\sigma_{i,p+1} \leq \hat{\sigma}_{i,p} + 1 \leq m$  by (13). 2)  $\sigma_{i,p} < m$  and  $\sigma_{j,p} = m$  for some  $j \in N_i(p)$ . For this case we obtain  $\hat{\sigma}_{i,p} = m$ ,  $x'_{i,p+1} = x^*$  by (10) (11). Since  $\|x^*\| \leq M_0 \leq M_m$ , then by (13)  $\sigma_{i,p+1} = \hat{\sigma}_{i,p} = m$ . Thus,  $\sigma_{i,p+1} \leq m$  when  $\sigma_{i,p} < m$ . Thereby (A.9) holds. This means that

$$\sigma_{i,p+1} = m + 1 \text{ only if } \sigma_{i,p} = m. \quad (\text{A.10})$$

By definition from  $\sigma_{p+1} = m+1$  we know that there exists some  $i_0 \in \mathcal{V}$  such that  $\sigma_{i_0,p+1} = m+1$ . Then  $\sigma_{i_0,p} = m$  by (A.10), and hence  $\hat{\sigma}_{i_0,p} = m$  from (A.4). Then from  $\sigma_{i_0,p+1} = m+1$  by (13) we derive  $\|x'_{i_0,p+1}\| > M_m$ , and hence  $\|\hat{x}_{i_0,p+1}\| = \|x'_{i_0,p+1}\| > M_m$  by (A.8). Then from (28) (29) we derive  $\hat{x}_{i,p+1} = x^* \quad \forall i \in \mathcal{V}$  and  $\sigma_{p+1} = m+1$ , which is consistent with  $\sigma_{p+1}$  defined by (6). Since  $\sigma_{p+1} = m+1$  and  $p \in [\tau_m, \tau_{m+1})$ , by (26) or from (21) (22), we see  $\hat{x}_{i,p+1} = x^* \quad \forall i \in \mathcal{V}$ . This is consistent with that produced by (27)-(29).

Case 2: We now consider the case  $\sigma_{p+1} = m$ . In this case,  $\sigma_{i,p+1} \leq m \quad \forall i \in \mathcal{V}$ . By (A.4), from (12) (13) we see

$$\|x'_{i,p+1}\| \leq M_m, \quad x_{i,p+1} = x'_{i,p+1} \quad \forall i : \sigma_{i,p} = m. \quad (\text{A.11})$$

So, by (A.8) we have

$$\|\hat{x}_{i,p+1}\| = \|x'_{i,p+1}\| \leq M_m \quad \forall i : \sigma_{i,p} = m. \quad (\text{A.12})$$

From  $\|x^*\| \leq M_0 \leq M_m$  and (A.3) we derive  $\|\hat{x}_{i,p+1}\| \leq M_m \quad \forall i : \sigma_{i,p} < m$ , which incorporating with (A.12) yields  $\|\hat{x}_{i,p+1}\| \leq M_m \quad \forall i \in \mathcal{V}$ . Then from (28) we have

$$\hat{x}_{i,p+1} = \hat{x}_{i,p+1} \quad \forall i \in \mathcal{V}, \quad \sigma_{p+1} = m. \quad (\text{A.13})$$

Thus  $\sigma_{p+1}$  is consistent with that defined by (6).

It remains to show that  $\hat{x}_{i,p+1}$  generated by (27)-(29) is consistent with that defined by (21) (22). We consider two cases: 1)  $\sigma_{i,p} = m$ . For this case, by (A.8) (A.11) (A.13) we see  $\hat{x}_{i,p+1} = x_{i,p+1} \quad \forall i : \sigma_{i,p} = m$ . By  $\sigma_{p+1} = m$  we see  $p+1 \in [\tau_m, \tau_{m+1})$ , and hence  $\hat{x}_{i,p+1} = x_{i,p+1}$  by (24). So, the assertion holds for any  $i$  with  $\sigma_{i,p} = m$ . 2)  $\sigma_{i,p} < m$ . From  $\sigma_{p+1} = m$  we see  $p+1 \in [\tau_m, \tau_{m+1})$ , and hence by  $\sigma_{i,p} < m$  from (25) we see that  $\hat{x}_{i,p+1}$  defined by (21) (22) equals  $x^*$ . By (A.3) (A.13) we derive  $\hat{x}_{i,p+1} = x^*$ , and hence the assertion holds for any  $i$  with  $\sigma_{i,p} < m$ .

In summary, we have shown that  $\hat{x}_{i,p+1}$  and  $\sigma_{p+1}$  generated by (27)-(29) are consistent with their definitions (21) (22) (6). This completes the proof.  $\blacksquare$

#### APPENDIX B PROOF OF LEMMA 4.5

For the lemma it suffices to prove

$$\begin{aligned} & \lim_{T \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{1}{T} \left\| \sum_{s=n_k}^{m(n_k, t_k) \wedge (\tau_{\sigma_{n_k}+1}-1)} \gamma_s \tilde{\varepsilon}_{i,s+1} I_{[\|\tilde{x}_{i,s}\| \leq K]} \right\| \\ & = 0 \quad \forall t_k \in [0, T] \text{ for sufficiently large } K > 0 \end{aligned} \quad (\text{B.1})$$

along indices  $\{n_k\}$  whenever  $\{\tilde{x}_{i,n_k}\}$  converges for the sample path  $\omega$  where A5 b) holds for agent  $i$ .

We consider the following two cases:

Case 1:  $\lim_{k \rightarrow \infty} \sigma_k = \sigma < \infty$ . We now show

$$\tau_{\sigma+1} = \infty \text{ when } \lim_{k \rightarrow \infty} \sigma_k = \sigma. \quad (\text{B.2})$$

Recall that  $\sigma_k$  is defined as the largest truncation number among all agents at time  $k$ , from  $\lim_{k \rightarrow \infty} \sigma_k = \sigma$  we have  $\sigma_{i,k} \leq \sigma \quad \forall k \geq 0 \quad \forall i \in \mathcal{V}$ . From here by the definition of  $\tau_{i,m}$  it follows that  $\tau_{i,\sigma+1} = \inf\{k : \sigma_{i,k} = \sigma+1\} = \infty \quad \forall i \in \mathcal{V}$ , and hence  $\tau_{\sigma+1} = \infty$ . Thus, (B.2) holds.

From (31) we have  $\tilde{\tau}_{i,\sigma} \leq \tau_{\sigma} + BD$ , and hence by (22) (B.2)

$$\tilde{x}_{i,k} = x_{i,k}, \quad \tilde{\varepsilon}_{i,k+1} = \varepsilon_{i,k+1} \quad \forall k \geq \tau_{\sigma} + BD. \quad (\text{B.3})$$

So,  $\left\| \sum_{s=k}^{m(k,t) \wedge (\tau_{\sigma_{k+1}}-1)} \gamma_s \tilde{\varepsilon}_{i,s+1} I_{[\|\tilde{x}_{i,s}\| \leq K]} \right\| = \left\| \sum_{s=k}^{m(k,t)} \gamma_s \varepsilon_{i,s+1} I_{[\|x_{i,s}\| \leq K]} \right\|$  for any  $t > 0$  and any sufficiently large  $k$ . Then we conclude (B.1) by A5 b).

Case 2:  $\lim_{k \rightarrow \infty} \sigma_k = \infty$ . We prove (B.1) separately for the following three cases:

i)  $\tilde{\tau}_{i,\sigma_{n_p}} \leq n_p$ . For this case,  $[n_p, \tau_{\sigma_{n_p}+1}) \subset [\tilde{\tau}_{i,\sigma_{n_p}}, \tau_{\sigma_{n_p}+1})$ , and hence from (22) we derive

$$\tilde{x}_{i,s} = x_{i,s}, \quad \tilde{\varepsilon}_{i,s+1} = \varepsilon_{i,s+1} \quad \forall s : n_p \leq s < \tau_{\sigma_{n_p}+1}. \quad (\text{B.4})$$

Thus, for sufficiently large  $K$  and any  $t_p \in [0, T]$

$$\begin{aligned} & \left\| \sum_{s=n_p}^{m(n_p, t_p) \wedge (\tau_{\sigma_{n_p}+1}-1)} \gamma_s \tilde{\varepsilon}_{i,s+1} I_{[\|\tilde{x}_{i,s}\| \leq K]} \right\| \\ & = \left\| \sum_{s=n_p}^{m(n_p, t_p) \wedge (\tau_{\sigma_{n_p}+1}-1)} \gamma_s \varepsilon_{i,s+1} I_{[\|x_{i,s}\| \leq K]} \right\|. \end{aligned} \quad (\text{B.5})$$

By (B.4) we conclude that  $\{x_{i,n_p}\}$  is a convergent subsequence. Noticing  $\sum_{s=n_p}^{m(n_p, t_p) \wedge (\tau_{\sigma_{n_p}+1}-1)} \gamma_s \leq \sum_{s=n_p}^{m(n_p, t_p)} \gamma_s \leq t_p \leq T$ , we then conclude (B.1) by (B.5) and A5 b).

ii)  $\tilde{\tau}_{i,\sigma_{n_p}} > n_p$  and  $\tilde{\tau}_{i,\sigma_{n_p}} = \tau_{\sigma_{n_p}+1}$ . By the definitions of  $\tau_k$  and  $\sigma_k$  we derive  $\tau_{\sigma_k} \leq k$ , and hence  $\tau_{\sigma_{n_p}} \leq n_p$ . Then  $[n_p, \tau_{\sigma_{n_p}+1}) \subset [\tau_{\sigma_{n_p}}, \tilde{\tau}_{i,\sigma_{n_p}})$ , and hence by (21) we have

$$\tilde{x}_{i,s} = x^*, \quad \tilde{\varepsilon}_{i,s+1} = -f_i(x^*) \quad \forall s : n_p \leq s < \tau_{\sigma_{n_p}+1}.$$

From  $\tilde{\tau}_{i,\sigma_{n_p}} = \tau_{\sigma_{n_p}+1}$  by (31) we see  $\tau_{\sigma_{n_p}+1} \leq \tau_{\sigma_{n_p}} + BD \leq n_p + BD$ . Then for sufficiently large  $K$  and any  $t_p \in [0, T]$

$$\begin{aligned} & \left\| \sum_{s=n_p}^{m(n_p, t_p) \wedge (\tau_{\sigma_{n_p}+1}-1)} \gamma_s \tilde{\varepsilon}_{i,s+1} I_{[\|\tilde{x}_{i,s}\| \leq K]} \right\| \\ & \leq \sum_{s=n_p}^{n_p+BD} \gamma_s \|f_i(x^*)\| \xrightarrow{p \rightarrow \infty} 0, \end{aligned}$$

and hence (B.1) holds.

iii)  $\tilde{\tau}_{i,\sigma_{n_p}} > n_p$  and  $\tilde{\tau}_{i,\sigma_{n_p}} < \tau_{\sigma_{n_p}+1}$ . By the definition of  $\tilde{\tau}_{i,\sigma_{n_p}}$  from  $\tilde{\tau}_{i,\sigma_{n_p}} < \tau_{\sigma_{n_p}+1}$  it follows that  $\tilde{\tau}_{i,\sigma_{n_p}} = \tau_{i,\sigma_{n_p}}$ , and hence by (31)  $\tau_{i,\sigma_{n_p}} \leq \tau_{\sigma_{n_p}} + BD$ . By noticing  $\tau_{\sigma_{n_p}} \leq n_p$  we conclude that

$$\tau_{\sigma_{n_p}} \leq n_p < \tilde{\tau}_{i,\sigma_{n_p}} = \tau_{i,\sigma_{n_p}} \leq n_p + BD. \quad (\text{B.6})$$

So,  $[n_p, \tau_{i,\sigma_{n_p}}) \subset [\tau_{\sigma_{n_p}}, \tilde{\tau}_{i,\sigma_{n_p}})$ . Then from here and  $\tilde{\tau}_{i,\sigma_{n_p}} = \tau_{i,\sigma_{n_p}}$  by (21) (22) we derive

$$\begin{aligned} \tilde{x}_{i,s} &= x^*, \quad \tilde{\varepsilon}_{i,s+1} = -f_i(x^*) \quad \forall s : n_p \leq s < \tau_{i,\sigma_{n_p}}, \\ \tilde{x}_{i,s} &= x_{i,s}, \quad \tilde{\varepsilon}_{i,s+1} = \varepsilon_{i,s+1} \quad \forall s : \tau_{i,\sigma_{n_p}} \leq s < \tau_{\sigma_{n_p}+1}. \end{aligned}$$

Consequently, for sufficiently large  $K$  and any  $t_p \in [0, T]$

$$\begin{aligned} & \left\| \sum_{s=n_p}^{m(n_p, t_p) \wedge (\tau_{\sigma_{n_p}+1}-1)} \gamma_s \tilde{\varepsilon}_{i,s+1} I_{[\|\tilde{x}_{i,s}\| \leq K]} \right\| \\ & \leq \left\| \sum_{s=n_p}^{m(n_p, t_p) \wedge (\tau_{\sigma_{n_p}+1}-1)} \gamma_s f_i(x^*) I_{[n_p \leq s < \tau_{i,\sigma_{n_p}}]} \right\| \\ & + \left\| \sum_{s=\tau_{i,\sigma_{n_p}}}^{m(n_p, t_p) \wedge (\tau_{\sigma_{n_p}+1}-1)} \gamma_s \varepsilon_{i,s+1} I_{[\|x_{i,s}\| \leq K]} \right\|. \end{aligned} \quad (\text{B.7})$$

Note that the first term at the right hand of (B.7) is smaller than  $\sum_{s=n_p}^{\tau_{i,\sigma_{n_p}}} \gamma_s \|f_i(x^*)\|$ , which tends to zero as  $k \rightarrow \infty$  by the last inequality in (B.6) and  $\lim_{k \rightarrow \infty} \gamma_k = 0$ . The truncation number for agent  $i$  at time  $\tau_{i,\sigma_{n_p}}$  is  $\sigma_{n_p}$ , while it is smaller than  $\sigma_{n_p}$  at time  $\tau_{i,\sigma_{n_p}} - 1$  since  $\tau_{i,\sigma_{n_p}}$  is the smallest time when the truncation number of  $i$  has reached  $\sigma_{n_p}$ . Consequently, by

Remark 3.1 we have  $x_{i,\tau_i,\sigma_{n_p}} = x^*$ , and hence  $\{x_{i,\tau_i,\sigma_{n_p}}\}_{p \geq 1}$  is a convergent subsequence. Noticing

$$\sum_{s=\tau_i,\sigma_{n_p}}^{m(n_p,t_p) \wedge (\tau_{\sigma_{n_p}+1}-1)} \gamma_s \leq \sum_{s=n_p}^{m(n_p,t_p)} \gamma_s \leq t_p,$$

we conclude (B.1) by (B.7) and A5 b).

Since one of i), ii), iii) must take place for the case  $\lim_{k \rightarrow \infty} \sigma_k = \infty$ , we thus have proved (B.1) in Case 2.

Combining Case 1 and Case 2 we derive (B.1). ■

## APPENDIX C PROOF OF LEMMA 4.6

Let us consider a fixed  $\omega$  where A5 b) holds.

Let  $C > \|\tilde{X}\|$ . There exists an integer  $k_C > 0$  such that

$$\|\tilde{X}_{n_k}\| \leq C \quad \forall k \geq k_C. \quad (\text{C.1})$$

By Lemma 4.5 we know that there exist a constant  $T_1 > 0$  and a positive integer  $k_0 \geq k_C$  such that

$$\left\| \sum_{s=n_k}^{m(n_k,t_k) \wedge (\tau_{\sigma_{n_k}+1}-1)} \gamma_s \tilde{\varepsilon}_{s+1} I_{[\|\tilde{X}_s\| \leq K]} \right\| \leq T_0 \quad (\text{C.2})$$

$$\forall t_k \in [0, T_0] \quad \forall T_0 \in [0, T_1] \quad \forall k \geq k_0$$

for sufficiently large  $K > 0$ . Define

$$M'_0 \triangleq 1 + C(c\rho + 2), \quad (\text{C.3})$$

$$H_1 \triangleq \max_X \{\|F(X)\| : \|X\| \leq M'_0 + 1 + C\}, \quad (\text{C.4})$$

$$c_1 \triangleq H_1 + 3 + \frac{c(\rho + 1)}{1 - \rho}, \quad \text{and} \quad c_2 \triangleq \frac{H_1 + 1}{\sqrt{N}}, \quad (\text{C.5})$$

where  $c$  and  $\rho$  are given by (16). Select  $T > 0$  such that

$$0 < T \leq T_1 \quad \text{and} \quad c_1 T < 1. \quad (\text{C.6})$$

For any  $k \geq k_0$  and any  $T_k \in [0, T]$  define

$$\begin{aligned} s_k &\triangleq \sup\{s \geq n_k : \|\tilde{X}_j - \tilde{X}_{n_k}\| \\ &\leq c_1 T_k + M'_0 \quad \forall j : n_k \leq j \leq s\}. \end{aligned} \quad (\text{C.7})$$

Then from (C.1) and (C.6) it follows that

$$\|\tilde{X}_s\| \leq M'_0 + 1 + C \quad \forall s : n_k \leq s \leq s_k. \quad (\text{C.8})$$

We intend to prove  $s_k > m(n_k, T_k)$ . Assume the converse that for sufficiently large  $k \geq k_0$  and any  $T_k \in [0, T]$

$$s_k \leq m(n_k, T_k). \quad (\text{C.9})$$

We first show that there exists a positive integer  $k_1 > k_0$  such that for sufficiently large  $k \geq k_1$

$$s_k < \tau_{\sigma_{n_k}+1} \quad \forall k \geq k_1 \quad \forall T_k \in [0, T]. \quad (\text{C.10})$$

We prove (C.10) for the two alternative cases:  $\lim_{k \rightarrow \infty} \sigma_k = \infty$  and  $\lim_{k \rightarrow \infty} \sigma_k = \sigma < \infty$ .

i)  $\lim_{k \rightarrow \infty} \sigma_k = \infty$ . Since  $\{M_k\}$  is a sequence of positive numbers increasingly diverging to infinity, there exists a positive integer  $k_1 > k_0$  such that  $M_{\sigma_{n_k}} > M'_0 + 1 + C$  for all  $k \geq k_1$ . Hence, from (C.8) we know  $s_k < \tau_{\sigma_{n_k}+1}$ .

ii)  $\lim_{k \rightarrow \infty} \sigma_k = \sigma < \infty$ . For this case there exists a positive integer  $k_1 > k_0$  such that  $\sigma_{n_k} = \sigma$  for all  $k \geq k_1$ , and hence  $\tau_{\sigma_{n_k}+1} = \tau_{\sigma+1} = \infty$  by (B.2). Then  $m(n_k, T_k) < \tau_{\sigma_{n_k}+1}$ , and hence by (C.9) we derive (C.10).

By (C.6) we see  $T_k \in [0, T] \subset [0, T_1]$ , then from (C.2) it follows that for sufficiently large  $k \geq k_1$  and any  $T_k \in [0, T]$

$$\left\| \sum_{s=n_k}^{m(n_k,t_k) \wedge (\tau_{\sigma_{n_k}+1}-1)} \gamma_s \tilde{\varepsilon}_{s+1} I_{[\|\tilde{X}_s\| \leq K]} \right\| \leq T_k \quad \forall t_k \in [0, T_k] \quad (\text{C.11})$$

for sufficiently large  $K > 0$ . By setting  $t_k = \sum_{m=n_k}^s \gamma_m$  for some  $s \in [n_k, s_k]$ , from (C.9) we see  $\sum_{m=n_k}^s \gamma_m \leq \sum_{m=n_k}^{s_k} \gamma_m \leq T_k$ . Noticing  $m(n_k, t_k) = s$ , from (C.10) we derive  $m(n_k, t_k) \wedge (\tau_{\sigma_{n_k}+1} - 1) = s$ . Then by setting  $K \triangleq M'_0 + 1 + C$ , from (C.8) (C.11) it follows that

$$\left\| \sum_{m=n_k}^s \gamma_m \tilde{\varepsilon}_{m+1} \right\| \leq T_k \quad \forall s : n_k \leq s \leq s_k \quad (\text{C.12})$$

for sufficiently large  $k \geq k_1$  and any  $T_k \in [0, T]$ .

Let us consider the following algorithm starting from  $n_k$  without truncation

$$Z_{m+1} = (W(m) \otimes \mathbf{I}_l) Z_m + \gamma_m (F(Z_m) + \tilde{\varepsilon}_{m+1}), \quad Z_{n_k} = \tilde{X}_{n_k}. \quad (\text{C.13})$$

By (C.10) we know that (37) holds for  $m = n_k, \dots, s_k - 1$  for  $\forall k \geq k_1 \quad \forall T_k \in [0, T]$ . Then from here we derive

$$Z_m = \tilde{X}_m \quad \forall m : n_k \leq m \leq s_k. \quad (\text{C.14})$$

Hence by (C.4) (C.8) we know that for  $\forall k \geq k_1 \quad \forall T_k \in [0, T]$

$$\|F(Z_m)\| \leq H_1 \quad \forall m : n_k \leq m \leq s_k. \quad (\text{C.15})$$

Set  $z_k = \frac{1^T \otimes \mathbf{I}_l}{N} Z_k$ . By multiplying both sides of (C.13) from left with  $\frac{1}{N} (\mathbf{1}^T \otimes \mathbf{I}_l)$ , from  $\mathbf{1}^T W(m) = \mathbf{1}^T$  and (34) we derive

$$z_{s+1} = z_s + \frac{\mathbf{1}^T \otimes \mathbf{I}_l}{N} \gamma_s (F(Z_s) + \tilde{\varepsilon}_{s+1}),$$

and hence

$$\begin{aligned} \|z_{s+1} - z_{n_k}\| &= \left\| \frac{\mathbf{1}^T \otimes \mathbf{I}_l}{N} \sum_{m=n_k}^s \gamma_m (F(Z_m) + \tilde{\varepsilon}_{m+1}) \right\| \\ &\leq \frac{1}{\sqrt{N}} \left( \sum_{m=n_k}^s \gamma_m \|F(Z_m)\| + \sum_{m=n_k}^s \gamma_m \|\tilde{\varepsilon}_{m+1}\| \right). \end{aligned}$$

Then from here by (C.9) (C.12) (C.15) we conclude that for sufficiently large  $k \geq k_1$  and any  $T_k \in [0, T]$

$$\|z_{s+1} - z_{n_k}\| \leq \frac{H_1 + 1}{\sqrt{N}} \sum_{m=n_k}^s \gamma_m = c_2 T_k \quad \forall s : n_k \leq s \leq s_k. \quad (\text{C.16})$$

Denote by  $Z_{\perp,s} \triangleq D_{\perp} Z_s$  the disagreement vector of  $Z_s$ . By multiplying both sides of (C.13) from left with  $D_{\perp}$  we derive

$$Z_{\perp,s+1} = D_{\perp} (W(s) \otimes \mathbf{I}_l) Z_s + \gamma_s D_{\perp} (F(Z_s) + \tilde{\varepsilon}_{s+1}),$$

and inductively

$$Z_{\perp,s+1} = \Psi(s, n_k) Z_{n_k} + \sum_{m=n_k}^s \gamma_m \Psi(s-1, m) D_{\perp} (F(Z_m) + \tilde{\varepsilon}_{m+1}) \quad \forall s \geq n_k.$$

From here by (35) (36) we derive

$$\begin{aligned} Z_{\perp,s+1} &= [(\Phi(s, n_k) - \frac{1}{N} \mathbf{1}\mathbf{1}^T) \otimes I_l] Z_{n_k} \\ &+ \sum_{m=n_k}^s \gamma_m [(\Phi(s-1, m) - \frac{1}{N} \mathbf{1}\mathbf{1}^T) \otimes I_l] F(Z_m) \\ &+ \sum_{m=n_k}^s \gamma_m [(\Phi(s-1, m) - \frac{1}{N} \mathbf{1}\mathbf{1}^T) \otimes I_l] \tilde{\varepsilon}_{m+1}. \end{aligned} \quad (\text{C.17})$$

From (C.1) (C.14) we have  $\|Z_{n_k}\| \leq C$ , and hence from (16) (C.15) it follows that

$$\begin{aligned} \|Z_{\perp,s+1}\| &\leq Cc\rho^{s+1-n_k} + \sum_{m=n_k}^s \gamma_m H_1 c\rho^{s-m} + \\ &\| \sum_{m=n_k}^s \gamma_m [(\Phi(s-1, m) - \frac{1}{N} \mathbf{1}\mathbf{1}^T) \otimes I_l] \tilde{\varepsilon}_{m+1} \| . \end{aligned} \quad (\text{C.18})$$

By (C.12) we derive  $\|\Gamma_s - \Gamma_{n_k-1}\| \leq T_k \quad \forall s : n_k \leq s \leq s_k$ , where  $\Gamma_n \triangleq \sum_{m=1}^n \gamma_m \tilde{\varepsilon}_{m+1}$ . Notice

$$\begin{aligned} &\sum_{m=n_k}^s \gamma_m (\Phi(s-1, m) \otimes I_l) \tilde{\varepsilon}_{m+1} \\ &= \sum_{m=n_k}^s (\Phi(s-1, m) \otimes I_l) (\Gamma_m - \Gamma_{m-1}) \\ &= \sum_{m=n_k}^s (\Phi(s-1, m) \otimes I_l) (\Gamma_m - \Gamma_{n_k-1}) \\ &- \sum_{m=n_k}^s (\Phi(s-1, m) \otimes I_l) (\Gamma_{m-1} - \Gamma_{n_k-1}). \end{aligned}$$

Summing by parts, by (16) we have

$$\begin{aligned} &\| \sum_{m=n_k}^s \gamma_m (\Phi(s-1, m) \otimes I_l) \tilde{\varepsilon}_{m+1} \| \leq \| \Gamma_s - \Gamma_{n_k-1} \| + \\ &\sum_{m=n_k}^{s-1} \| \Phi(s-1, m) - \Phi(s-1, m+1) \| \| \Gamma_m - \Gamma_{n_k-1} \| \\ &\leq T_k + \sum_{m=n_k}^{s-1} (c\rho^{s-m-1} + c\rho^{s-m}) T_k \\ &\leq T_k + \frac{c(\rho+1)}{1-\rho} T_k \quad \forall s : n_k \leq s \leq s_k, \end{aligned}$$

which incorporating with (C.12) yields

$$\begin{aligned} &\| \sum_{m=n_k}^s \gamma_m [(\Phi(s-1, m) - \frac{1}{N} \mathbf{1}\mathbf{1}^T) \otimes I_l] \tilde{\varepsilon}_{m+1} \| \\ &\leq (2 + \frac{c(\rho+1)}{1-\rho}) T_k \quad \forall s : n_k \leq s \leq s_k \end{aligned} \quad (\text{C.19})$$

for sufficiently large  $k \geq k_1$  and any  $T_k \in [0, T]$ .

By noticing  $\sum_{m=n_k}^s \gamma_m \rho^{s-m} \leq \frac{1}{1-\rho} \sup_{m \geq n_k} \gamma_m$ , from  $\lim_{k \rightarrow \infty} \gamma_k = 0$  and (C.18) (C.19) it is concluded that

$$\begin{aligned} \|Z_{\perp,s+1}\| &\leq Cc\rho + \frac{cH_1}{1-\rho} \sup_{m \geq n_k} \gamma_m + (2 + \frac{c(\rho+1)}{1-\rho}) T_k \\ &\leq Cc\rho + 1 + (2 + \frac{c(\rho+1)}{1-\rho}) T_k \quad \forall s : n_k \leq s \leq s_k \end{aligned} \quad (\text{C.20})$$

for sufficiently large  $k \geq k_1$  and any  $T_k \in [0, T]$ . By noticing  $Z_s = Z_{\perp,s} + (\mathbf{1} \otimes I_l) z_s$ , we derive

$$\begin{aligned} &\|Z_{s+1} - Z_{n_k}\| \\ &= \|(\mathbf{1} \otimes I_l) z_{s+1} + Z_{\perp,s+1} - Z_{\perp,n_k} - (\mathbf{1} \otimes I_l) z_{n_k}\| \\ &\leq \|Z_{\perp,s+1}\| + \|Z_{\perp,n_k}\| + \sqrt{N} \|z_{s+1} - z_{n_k}\|. \end{aligned}$$

Since  $\|Z_{\perp,n_k}\| \leq 2 \|Z_{n_k}\| = 2C$ , from (C.16) (C.20) it follows that for sufficiently large  $k \geq k_1$  and any  $T_k \in [0, T]$

$$\begin{aligned} \|Z_{s+1} - Z_{n_k}\| &\leq Cc\rho + 1 + (2 + \frac{c(\rho+1)}{1-\rho}) T_k + 2C + \\ &(H_1 + 1) T_k \leq C(c\rho + 2) + 1 + (3 + H_1 + \frac{c(\rho+1)}{1-\rho}) T_k \\ &= M'_0 + c_1 T_k \quad \forall s : n_k \leq s \leq s_k. \end{aligned} \quad (\text{C.21})$$

Therefore, from (C.6) and  $\|Z_{n_k}\| \leq C$  we know that for sufficiently large  $k \geq k_1$  and any  $T_k \in [0, T]$

$$\|Z_{s_k+1}\| \leq \|Z_{n_k}\| + M'_0 + c_1 T_k \leq M'_0 + 1 + C.$$

Rewrite (27) in the compact form as follows

$$\hat{X}_{s_k+1} = [W(s_k) \otimes I_l] \tilde{X}_{s_k} + \gamma_{s_k} (F(\tilde{X}_{s_k}) + \tilde{\varepsilon}_{s_k+1}),$$

where  $\hat{X}_k \triangleq \text{col}\{\hat{x}_{1,k}, \dots, \hat{x}_{N,k}\}$ . Then by (C.13) (C.14)  $\hat{X}_{s_k+1} = Z_{s_k+1}$ , and hence from (C.22) it follows that

$$\|\hat{X}_{s_k+1}\| \leq M'_0 + 1 + C. \quad (\text{C.22})$$

We now show

$$\tilde{X}_{s_k+1} = \hat{X}_{s_k+1} \text{ and } s_k + 1 < \tau_{\sigma_{n_k}+1} \quad (\text{C.23})$$

for sufficiently large  $k \geq k_1$  and any  $T_k \in [0, T]$ . We consider the following two cases:  $\lim_{k \rightarrow \infty} \sigma_k = \infty$  and  $\lim_{k \rightarrow \infty} \sigma_k = \sigma < \infty$ .

i)  $\lim_{k \rightarrow \infty} \sigma_k = \infty$ . By noting  $M_{\sigma_{n_k}} > M'_0 + 1 + C \quad \forall k \geq k_1$ , from (C.22) by (28) (29) we see  $\tilde{X}_{s_k+1} = \hat{X}_{s_k+1}$  and  $\sigma_{s_k+1} = \sigma_{s_k}$ . Hence  $s_k + 1 < \tau_{\sigma_{n_k}+1}$  by (C.10).

ii)  $\lim_{k \rightarrow \infty} \sigma_k = \sigma < \infty$ . Since  $\tau_{\sigma_{n_k}+1} = \infty \quad \forall k \geq k_1$ , by (C.9) we see  $s_k + 1 < \tau_{\sigma_{n_k}+1}$ . Then by  $\sigma_{n_k} = \sigma \quad \forall k \geq k_1$  we conclude  $\sigma_{s_k+1} = \sigma_{s_k} = \sigma$ . Hence by (28) we derive  $\tilde{X}_{s_k+1} = \hat{X}_{s_k+1}$ . Thus (C.23) holds.

From (C.23) we know that (37) holds for  $m = s_k$  for sufficiently large  $k \geq k_1$  and any  $T_k \in [0, T]$ . From  $\hat{X}_{s_k+1} = Z_{s_k+1}$  by (C.23) we see  $\tilde{X}_{s_k+1} = Z_{s_k+1}$ . Hence from (C.21) and  $Z_{n_k} = \tilde{X}_{n_k}$  it follows that for sufficiently large  $k \geq k_1$  and any  $T_k \in [0, T]$

$$\|\tilde{X}_{s_k+1} - \tilde{X}_{n_k}\| \leq M'_0 + c_1 T_k,$$

which contradicts with the definition of  $s_k$  given by (C.7). Thus (C.9) does not hold. So,  $s_k > m(n_k, T_k)$  and hence by the definition of  $s_k$  given in (C.7) we derive (38).

Since  $\{\tilde{X}_s : n_k \leq s \leq m(n_k, T_k)\}$  are bounded, similar to proving (C.10) it can be shown that  $m(n_k, T_k) + 1 < \tau_{\sigma_{n_k}+1}$ . Thus (37) holds for  $m = n_k, \dots, m(n_k, T_k)$ . Similar to (C.16) we obtain

$$\|\bar{x}_{m+1} - \bar{x}_{n_k}\| \leq c_2 T_k \quad \forall m : n_k \leq m \leq m(n_k, T_k)$$

for sufficiently large  $k$  and any  $T_k \in [0, T]$ . Hence, (39) holds.

The proof of Lemma 4.6 is completed.  $\blacksquare$

## APPENDIX D PROOF OF LEMMA 4.7

Since  $\lim_{k \rightarrow \infty} \tilde{X}_{n_k} = \bar{X}$ , by setting  $\bar{x} \triangleq \frac{\mathbf{1}^T \otimes I_l}{N} \bar{X}$  we then have  $\lim_{k \rightarrow \infty} \bar{x}_{n_k} = \bar{x}$ . Lemma 4.6 ensures that there exists a  $T \in (0, 1)$  such that  $m(n_k, T) < \tau_{\sigma_{n_k}+1}$  and  $\{\tilde{X}_s : n_k \leq s \leq m(n_k, T) + 1\}$  are bounded for sufficiently large  $k$ . So, for any  $T_k \in [0, T]$  and any sufficiently large  $K$

$$\| \sum_{s=n_k}^{m(n_k, T_k) \wedge (\tau_{\sigma_{n_k}+1}-1)} \gamma_s \tilde{\varepsilon}_{s+1} I_{[\|\tilde{X}_s\| \leq K]} \| = \| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s \tilde{\varepsilon}_{s+1} \|.$$

Therefore, by Lemma 4.5 we derive

$$\lim_{T \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{1}{T} \| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s \tilde{\varepsilon}_{s+1} \| = 0 \quad \forall T_k \in [0, T].$$

So, for (42) it suffices to show

$$\lim_{T \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{1}{T} \| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s e_{s+1} \| = 0 \quad \forall T_k \in [0, T]. \quad (\text{D.1})$$

Similar to (C.20), we can show that there exist positive constants  $c_3, c_4, c_5$  such that for sufficiently large  $k$

$$\| \tilde{X}_{\perp, s+1} \| \leq c_3 \rho^{s+1-n_k} + c_4 \sup_{m \geq n_k} \gamma_m + c_5 T \quad \forall s : n_k \leq s \leq m(n_k, T). \quad (\text{D.2})$$

Since  $0 < \rho < 1$ , there exists a positive integer  $m'$  such that  $\rho^{m'} < T$ . Then  $\sum_{m=n_k}^{n_k+m'} \gamma_m \xrightarrow{k \rightarrow \infty} 0$  by  $\lim_{k \rightarrow \infty} \gamma_k = 0$ . Thus,  $n_k + m' < m(n_k, T)$  for sufficiently large  $k$ . Therefore, from (D.2) we know that for sufficiently large  $k$

$$\| \tilde{X}_{\perp, s+1} \| \leq o(1) + (c_3 + c_5)T \quad \forall s : n_k + m' \leq s \leq m(n_k, T), \quad (\text{D.3})$$

where  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$ .

Since  $\tilde{x}_{n_k} \xrightarrow{k \rightarrow \infty} \bar{x}$ , from (39) (D.3) it follows that for sufficiently large  $k$  and  $\forall s : n_k + m' \leq s \leq m(n_k, T)$

$$\begin{aligned} \| \bar{x}_s - \bar{x} \| &\leq \| \bar{x}_s - \tilde{x}_{n_k} \| + \| \tilde{x}_{n_k} - \bar{x} \| = o(1) + \delta(T), \\ \| \tilde{x}_{i,s} - \bar{x} \| &\leq \| \tilde{x}_{i,s} - \tilde{x}_s \| + \| \tilde{x}_s - \bar{x} \| = o(1) + \delta(T), \end{aligned}$$

where  $\delta(T) \rightarrow 0$  as  $T \rightarrow 0$ . By continuity of  $f_i(\cdot)$  we derive  $\| f_i(\tilde{x}_{i,s}) - f_i(\tilde{x}_s) \| \leq \| f_i(\tilde{x}_{i,s}) - f_i(\bar{x}) \| + \| f_i(\tilde{x}_s) - f_i(\bar{x}) \| = o(1) + \delta(T)$ . Consequently,

$$\| e_{i,s+1} \| = \| f_i(\tilde{x}_{i,s}) - f_i(\tilde{x}_s) \| / N = o(1) + \delta(T).$$

Then for sufficiently large  $k$

$$\| e_{s+1} \| = o(1) + \delta(T) \quad \forall s : n_k + m' \leq s \leq m(n_k, T). \quad (\text{D.4})$$

By the boundedness of  $\{\tilde{X}_s : n_k \leq s \leq m(n_k, T)\}$  and by continuity of  $f_i(\cdot)$  we know that there exists a constant  $c_e > 0$  such that  $\| e_{s+1} \| \leq c_e$ . Then from (D.4) we derive

$$\begin{aligned} &\sum_{s=n_k}^{m(n_k, T_k)} \gamma_s e_{s+1} \| \\ &\leq \sum_{s=n_k}^{n_k+m'} \gamma_s c_e + (o(1) + \delta(T)) \sum_{s=n_k+m'+1}^{m(n_k, T_k)} \gamma_s \\ &\leq c_e m' \sup_{s \geq n_k} \gamma_s + T(\delta(T) + o(1)) \quad \forall T_k \in [0, T] \end{aligned}$$

for sufficiently large  $k$ . From here by  $\lim_{k \rightarrow \infty} \gamma_k = 0$  we derive

$$\limsup_{k \rightarrow \infty} \frac{1}{T} \left\| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s e_{s+1} \right\| = \delta(T) \quad \forall T_k \in [0, T].$$

Letting  $T \rightarrow 0$ , we derive (D.1), and hence (42) holds.

The proof of Lemma 4.7 is completed.  $\blacksquare$

#### APPENDIX E PROOF OF LEMMA 4.9

i) Assume

$$\lim_{k \rightarrow \infty} \sigma_k = \infty. \quad (\text{E.1})$$

Then there exists a positive integer  $n_k$  such that  $\sigma_{n_k} = k$  and  $\sigma_{n_k-1} = k-1$  for any  $k \geq 1$ . Since  $n_k - 1 \in [\tau_{k-1}, \tau_k)$  from  $\sigma_{n_k} = k$ , by (26) we see  $\tilde{x}_{i,n_k} = x^* \quad \forall i \in \mathcal{V}$ . Consequently,  $\tilde{X}_{n_k} = (1 \otimes I_l)x^*$  and hence  $\{\tilde{X}_{n_k}\}$  is a convergent subsequence with  $\bar{x}_{n_k} = x^*$ .

Since  $\{M_k\}$  is a sequence of positive numbers increasingly diverging to infinity, there exists a positive integer  $k_0$  such that

$$M_k \geq 2\sqrt{N}c_0 + 2 + M'_1 \quad \forall k \geq k_0, \quad (\text{E.2})$$

where  $c_0$  is given in A2 c) and

$$M'_1 = 2 + (2\sqrt{N}c_0 + 2)(c\rho + 2). \quad (\text{E.3})$$

In what follows, we show that under the converse assumption  $\{\tilde{x}_{n_k}\}$  starting from  $x^*$  crosses the sphere with  $\|x\| = c_0$  infinitely many times. Define

$$m_k \triangleq \inf\{s > n_k : \|\tilde{X}_s\| \geq 2\sqrt{N}c_0 + 2 + M'_1\}, \quad (\text{E.4})$$

$$l_k \triangleq \sup\{s < m_k : \|\tilde{X}_s\| \leq 2\sqrt{N}c_0 + 2\}. \quad (\text{E.5})$$

Noticing  $\|\tilde{X}_{n_k}\| = \sqrt{N}\|x^*\|$  and  $\|x^*\| < c_0$ , we derive  $\|\tilde{X}_{n_k}\| < \sqrt{N}c_0$ . Hence from (E.4) (E.5) it is seen that  $n_k < l_k < m_k$ . By the definition of  $l_k$  we know that  $\{\tilde{X}_{l_k}\}$  is bounded, then there exists a convergent subsequence, denoted still by  $\{\tilde{X}_{l_k}\}$ . By denoting  $\bar{X}$  the limiting point of  $\tilde{X}_{l_k}$ , from  $\|\tilde{X}_{l_k}\| \leq 2\sqrt{N}c_0 + 2$  it follows that  $\|\bar{X}\| \leq 2\sqrt{N}c_0 + 2$ .

By Lemma 4.6 there exist constants  $M'_0 > 0$  defined by (C.3) with  $C = 2\sqrt{N}c_0 + 2$ ,  $c_1 > 0$  and  $c_2 > 0$  defined by (C.5),  $0 < T < 1$  with  $c_1 T \leq 1$  such that

$$\|\tilde{X}_{m+1} - \tilde{X}_{l_k}\| \leq c_1 T + M'_0 \quad \forall m : l_k \leq m \leq m(l_k, T)$$

for sufficiently large  $k \geq k_0$ . Then for sufficiently large  $k \geq k_0$

$$\begin{aligned} \|\tilde{X}_{m+1}\| &\leq \|\tilde{X}_{l_k}\| + c_1 T + M'_0 \\ &\leq 2\sqrt{N}c_0 + 2 + 1 + 1 + (2\sqrt{N}c_0 + 2)(c\rho + 2) \\ &= 2\sqrt{N}c_0 + 2 + M'_1 \quad \forall m : l_k \leq m \leq m(l_k, T). \end{aligned} \quad (\text{E.6})$$

Then  $m(l_k, T) \leq n_{k+1}$  for sufficiently large  $k \geq k_0$  by (E.2).

From (E.6) by the definition of  $m_k$  defined in (E.4), we conclude  $m(l_k, T) + 1 \leq m_k$  for sufficiently large  $k \geq k_0$ . Then by (E.4) (E.5) we know that for sufficiently large  $k \geq k_0$

$$2\sqrt{N}c_0 + 2 < \|\tilde{X}_{m+1}\| \leq 2\sqrt{N}c_0 + 2 + M'_1 \quad \forall m : l_k \leq m \leq m(l_k, T). \quad (\text{E.7})$$

Since  $0 < \rho < 1$ , there exists a positive integer  $m_0$  such that  $4c\rho^{m_0} < 1$ . Then  $\sum_{m=l_k}^{l_k+m_0} \gamma_m \xrightarrow{k \rightarrow \infty} 0$  by  $\gamma_k \xrightarrow{k \rightarrow \infty} 0$ , and hence  $l_k + m_0 < m(l_k, T) < n_{k+1}$  for sufficiently large  $k \geq k_0$ . So, from (E.7) it is seen that for sufficiently large  $k \geq k_0$

$$\|\tilde{X}_{l_k+m_0}\| > 2\sqrt{N}c_0 + 2. \quad (\text{E.8})$$

Noticing that  $\{\tilde{X}_{m+1} : l_k \leq m \leq m(l_k, T)\}$  are bounded, similarly to (C.20) we know that for sufficiently large  $k \geq k_0$

$$\begin{aligned} \|\tilde{X}_{\perp, m+1}\| &\leq (2\sqrt{N}c_0 + 2)c\rho^{m+1-l_k} + \frac{cH_1}{1-\rho} \sup_{m \geq l_k} \gamma_m \\ &\quad + 2T + \frac{c(\rho+1)}{1-\rho}T \quad \forall m : l_k \leq m \leq m(l_k, T). \end{aligned}$$

From here, by  $c_1 T < 1$  and  $\gamma_k \xrightarrow{k \rightarrow \infty} 0$  it follows that

$$\begin{aligned} \|\tilde{X}_{\perp, l_k+m_0}\| &\leq 2c\rho^{m_0}(\sqrt{N}c_0 + 1) + \frac{1}{2} + c_1 T \\ &\leq \frac{1}{2}(\sqrt{N}c_0 + 1) + \frac{1}{2} + 1 = \frac{\sqrt{N}c_0}{2} + 2 \end{aligned} \quad (\text{E.9})$$



for sufficiently large  $k \geq k_0$ . By noticing  $(\mathbf{1} \otimes \mathbf{I}_l)\bar{x}_{l_k+m_0} = \bar{X}_{l_k+m_0} - \bar{X}_{\perp, l_k+m_0}$ , from (E.8) (E.9) we conclude that

$$\begin{aligned} \sqrt{N} \|\bar{x}_{l_k+m_0}\| &= \|\bar{X}_{l_k+m_0} - \bar{X}_{\perp, l_k+m_0}\| \\ &\geq \|\bar{X}_{l_k+m_0}\| - \|\bar{X}_{\perp, l_k+m_0}\| > \frac{3}{2}\sqrt{N}c_0. \end{aligned} \quad (\text{E.10})$$

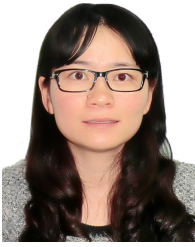
Therefore,  $\|\bar{x}_{l_k+m_0}\| > c_0$ .

Thus, we have shown that for sufficiently large  $k \geq k_0$ , starting from  $x^*$ ,  $\{\bar{x}_{n_k}\}$  crosses the sphere with  $\|x\| = c_0$  before the time  $n_{k+1}$ . So,  $\{\bar{x}_{n_k}\}$  starting from  $x^*$  crosses the sphere with  $\|x\| = c_0$  infinitely many times.

ii) Since  $v(J)$  is nowhere dense, there exists a nonempty interval  $[\delta_1, \delta_2] \in (v(x^*), \inf_{\|x\|=c_0} v(x))$  with  $d([\delta_1, \delta_2], v(J)) > 0$ . Assume the converse  $\lim_{k \rightarrow \infty} \sigma_k = 0$ . By i)  $\{v(\bar{x}_k)\}$  crosses the interval  $[\delta_1, \delta_2]$  infinitely many times while  $\{\bar{X}_{n_k}\}$  converges. This contradicts Lemma 4.8. Therefore, (E.1) is impossible and (53) holds. The proof is completed. ■

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